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Scattering of plane wave by circular-arc alluvial valley in a poroelastic half-space

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Abstract

Based on the complex variable function method, a new approach for solving the scattering of plane P wave by circular-arc alluvial valley in poroelastic half-space is developed in the paper. In this analysis, the poroelastic half-space and the circular-arc valley are modeled as poroelastic medium based on Biot's dynamic theory. By introducing three potentials, the governing equations for Biot's theory are reduced to three Helmholtz equations. The series solutions of the Helmholtz equations are obtained by the wave function expansion method. Here, the large circle assumption is applied to simulate the boundary conditions at the half-space boundary. The stresses and pore pressures are obtained by using the boundary conditions and continuous conditions of the poroelastic half-space and the circular-arc alluvial valley. Numerical results show that the dynamic stresses concentration and pore pressures concentration are mainly relative to the wave shape of incidence, angle of incidence, dimensionless frequency of incident wave, stiffness and pore ratio of the poroelastic half-space and valley.

1. Introduction

The analysis of the scattering of elastic waves by finite sized, surface irregularities including canyons, alluvial valleys and sedimentary basins is of great importance for civil engineering and earthquake engineering.

The scattering of elastic waves by cavities have been studied for a very long time. The most important contributions of which are summarized in the two well-known works [1,2]. There are many kinds of analytical and numerical methods that can be used to solve the dynamics response of cavities in elastic half-space. For example, Gamer [3] used wave function expansion method to study dynamic stress concentration factor at the surface of a semi-circular cavity in an elastic half-space. Bard and Bouchon [4,5] studied alluvial valleys by using discrete wavenumber approximations. Zeng and Cakmak [6] used series expansion method to investigate the scattering of SH waves by multiple cavities in both an infinite and a half-space. Davis et al. [7] used

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Fourier–Bessel series to investigate the transverse response of underground cylindrical cavities to incident shear waves. Sancar and Pao [8] gave solutions for the scattering of plane harmonic pressure waves by two cylindrical cavities by using eigenfunction expansion methods. Datta et al. [9] used combined finite element method and eigenfunction expansions method to study dynamic stresses and displacements around cylindrical cavities of various shapes in elastic medium. Also, there are methods based on the boundary element method [10–14]. For example, Moeen-Vaziri and Trifunac [15] used the boundary method to solve the problem of scattering and diffraction of SH waves by cylindrical canals of arbitrary shape in an elastic half-space.

However, the research concerning the scattering of elastic waves by cavities has been mainly restricted to the elastic case. For saturated porous medium, several scholars also have addressed the scattering of elastic waves by embedded cavities. For example, Mei et al. [16] introduced boundary layer approximation to study the scattering of P and SV waves by a circular cavity of arbitrary radius in a poroelastic medium. Norris [17] obtained the solution for a point load in an unbounded fluid saturated porous solid. Krutin et al. [18] solved the problem of elastic harmonic wave by a fluid filled cylindrical cavity embedded in saturated medium. Zimmerman [19] used boundary element method to study the problem of wave diffraction by a spherical cavity in an infinite poroelastic medium. Hu et al. [20] used Biot's theory to study the scattering and refraction of plane strain wave by a cylindrical cavity in a fluid saturated soil. Kumar et al. [21,22] obtained general solution of an anisotropic saturated poroelastic medium by using the Fourier transform and eigenvalue approach. Liang et al. [23] used wave function expansion method to obtain an analytical solution for the scattering of incident plane SV wave by a shallow circular-arc canyon in a saturated half-space. Kattis et al. [24] used boundary element method to solve the problem of P and SV waves by tunnel in an infinite poroelastic saturated soil. Wang et al. [25] used potential function and complex function to solve the problem of the scattering of plane wave by multiple elliptic cavities in saturated soil medium. Li et al. [26] obtained an analytical solution for scattering and diffraction of P wave by circular-arc alluvial valley with shallow saturated soil deposit. Hasheminejad and Avazmohammadi [27] investigated the dynamic response of plane wave with a pair of parallel circular cylindrical cavities buried in a boundless porous medium.

The purpose of the present study is to develop a new method for addressing the scattering of elastic wave by circular-arc alluvial valley in poroelastic half-space. The poroelastic medium is described by Biot's theory [28,29]. By introducing three potentials, the governing equations for Biot's theory are decoupled and reduced to three Helmholtz equations satisfied by three potentials. The series solutions for the Helmholtz equations are obtained by wave function expansion method. To illustrate the result, the effects of wave shape of incidence, angle of incidence, dimensionless frequency of incident P wave, stiffness and pore ratio of the poroelastic half-space and alluvial valley are studied. The methodology and analytical solution developed in this paper may provide a new method for further analysis of the scattering of transient wave by the irregular topography condition in a finite half-space.

2. Governing equations

The model involves circular-arc alluvial valley overlying a poroelastic half-space. Suppose the origin of the circular-arc alluvial valley is o_1, inner radius is r_4, outer radius is r_1, and depth is d_1 (see Fig. 1). The poroelastic half-space and circular-arc alluvial valley are described by Biot's theory. The constitutive equations for homogeneous poroelastic medium can be expressed as [28,29]

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}e - \alpha\delta_{ij}p_f \tag{1a}$$

$$p_f = -\alpha M e + M\vartheta \tag{1b}$$

$$e = u_{i,i} \tag{1c}$$

$$\vartheta = -w_{i,i} \tag{1d}$$

where σ_{ij} denotes the stress of bulk material; ε_{ij} and e are the strain component and the dilatation of the solid skeleton, respectively; λ, μ represent Lamé constants; δ_{ij} denotes the Kronecker delta; ϑ denotes the variation

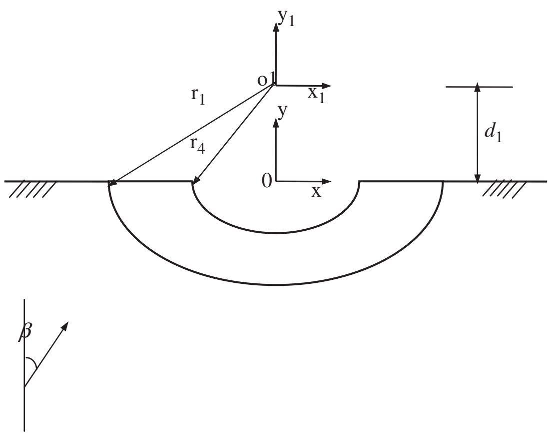

Fig. 1. Model of circular-arc alluvial valley in poroelastic half-space.

of fluid content per unit reference volume; α, M are Biot parameters; p_f is the pore pressure; u_i and w_i denote the average solid displacement and the fluid displacement relative to the solid frame.

The equations of motion for the poroelastic medium are expressed in terms of the displacement u_t and w_i as

$$\mu u_{i,jj} + (\lambda + \alpha^2 M + \mu)u_{j,ji} + \alpha M w_{j,ji} = \rho \ddot{u}_i + \rho_f \ddot{w}_i \tag{2a}$$

$$\alpha M u_{j,ji} + M w_{j,ji} = \rho_f \ddot{u}_i + m\ddot{w}_i + \frac{\eta}{k}\dot{w}_i \tag{2b}$$

where ρ, ρ_f denote the bulk density of the poroelastic medium and the density of the pore fluid, $\rho = (1-n)\rho_s + n\rho_f$, ρ_s is the density of the solid skeleton and n is the porosity of the poroelastic medium; $m = a_\infty \rho_f / n$ and a_∞ is tortuosity; η, k represent the viscosity and the permeability of the poroelastic medium, respectively; a superimposed dot on a variable denotes the derivative with respect to time.

In order to decouple the equations of motion for the poroelastic medium, two scalar potentials φ_f, φ_s and one vector potential $\boldsymbol{\psi}$ are introduced to express the displacement and the pore pressure. The displacement and the pore pressure are expressed by the potentials in the following form [19]:

$$u_i = \varphi_{,i} + e_{ijk}\boldsymbol{\psi}_{k,j} = \varphi_{f,i} + \varphi_{s,i} + e_{ijk}\boldsymbol{\psi}_{k,j} \tag{3a}$$

$$p_f = A_f \varphi_{f,ii} + A_s \varphi_{s,ii} \tag{3b}$$

where A_f and A_s are two constants to be determined by the governing equations of Biot's theory; e_{ijk} denotes the Levi–Civita symbol.

When considering the time harmonic vibration of frequency ω by the term $e^{-i\omega t}$, where $i = \sqrt{-1}$, for brevity, the term $e^{-i\omega t}$ is suppressed henceforth from all expressions in the sequel. Substituting Eqs. (1b), (3a) and (3b) into Eq. (2a), the following formula is obtained:

$$[(\lambda + 2\mu - \beta_2 A_f)\varphi_{f,jj} + \beta_3 \varphi_f]_{,i} + [(\lambda + 2\mu - \beta_2 A_s)\varphi_{s,jj} + \beta_3 \varphi_s]_{,i} + e_{iml}[\mu \boldsymbol{\psi}_{l,jj} + \beta_3 \boldsymbol{\psi}_l]_{,m} = 0 \tag{4}$$

In order to satisfy Eq. (4), the expressions in braces should be equal to zero independently. Thus, Eq. (4) can be written in the following form:

$$(\lambda + 2\mu - \beta_2 A_f)\varphi_{f,jj} + \beta_3 \varphi_f = 0 \tag{5a}$$

$$(\lambda + 2m - \beta_2 A_s)\varphi_{s,jj} + \beta_3 \varphi_s = 0 \tag{5b}$$

$$\mu \boldsymbol{\psi}_{i,jj} + \beta_3 \boldsymbol{\psi}_i = 0 \tag{5c}$$

where

$$\beta_3 = \rho\omega^2 + \rho_f^2 \omega^4/\beta_1, \quad \beta_2 = \alpha + \rho_f \omega^2/\beta_1, \quad \beta_1 = -m\omega^2 - i\eta\omega/k \tag{6}$$

Substituting Eq. (1b) into Eq. (2b) leads to

$$p_{f,ii} - p_f\beta_1/M - (\alpha\beta_1 + \rho_f\omega^2)u_{i,i} = 0 \tag{7}$$

Then, substituting Eq. (3) into Eq. (7) yields

$$[A_f\varphi_{f,ii} + (\beta_5 A_f - \beta_4)\varphi_f]_{,jj} + [A_s\varphi_{s,ii} + (\beta_5 A_s - \beta_4)\varphi_s]_{,jj} = 0 \tag{8}$$

In order to satisfy Eq. (8), the expressions in braces should be equal to zero independently. Thus, one has

$$A_f\varphi_{f,ii} + (\beta_5 A_f - \beta_4)\varphi_f = 0 \tag{9a}$$

$$A_s\varphi_{s,ii} + (\beta_5 A_s - \beta_4)\varphi_s = 0 \tag{9b}$$

$$\beta_4 = \alpha\beta_1 + \rho_f\omega^2 \tag{9c}$$

$$\beta_5 = -\beta_1/M \tag{9d}$$

From Eqs. (5a), (5b) and (9a), (9b), one has

$$A_{f,s}^2 + \frac{\beta_3 - (\lambda + 2\mu)\beta_5 - \beta_2\beta_4}{\beta_2\beta_5} A_{f,s} + \frac{(\lambda + 2\mu)\beta_4}{\beta_2\beta_5} = 0 \tag{10}$$

From Eqs. (5) and (9), each component $\varphi_{f,s}$ and $\boldsymbol{\psi}$ must satisfy Helmholtz equations of the following form:

$$\nabla^2\varphi_f + k_f^2\varphi_f = 0 \tag{11a}$$

$$\nabla^2\varphi_s + k_s^2\varphi_s = 0 \tag{11b}$$

$$\nabla^2\boldsymbol{\psi} + k_t^2\boldsymbol{\psi} = 0 \tag{11c}$$

If introducing

$$k_f^2 = \beta_3/(\lambda + 2\mu - \beta_2 A_f) = (\beta_5 A_f - \beta_4)/A_f \tag{12a}$$

$$k_s^2 = \beta_3/(\lambda + 2\mu - \beta_2 A_s) = (\beta_5 A_s - \beta_4)/A_s \tag{12b}$$

$$k_t^2 = \beta_3/\mu \tag{12c}$$

where k_f, k_s, k_t are the complex wavenumbers associated with the fast wave, slow wave, and shear wave, respectively. In order to guarantee the attenuation of the body waves, $\mathrm{Im}(k_f)$, $\mathrm{Im}(k_s)$, $\mathrm{Im}(k_t)$ should be non-positive. Also, since the speed of the fast wave is larger than that of the slow wave, as the result, the inequality $\mathrm{Re}(k_f) < \mathrm{Re}(k_s)$ should always hold.

If introducing complex variables $z = x + \mathrm{i}y$, $\bar{z} = x - \mathrm{i}y$, the general solutions of Eq. (11) can be expressed in terms of Hankel function

$$\varphi_f = \sum_{n=-\infty}^{\infty} a_n H_n^{(1)}(k_f|z|)\left(\frac{z}{|z|}\right)^n \tag{13a}$$

$$\varphi_s = \sum_{n=-\infty}^{\infty} b_n H_n^{(1)}(k_s|z|)\left(\frac{z}{|z|}\right)^n \tag{13b}$$

$$\boldsymbol{\psi} = \sum_{n=-\infty}^{\infty} c_n H_n^{(1)}(k_t|z|)\left(\frac{z}{|z|}\right)^n \tag{13c}$$

where $H_n^{(1)}(*)$ denotes the first kind of Hankel function of order n; a_n, b_n, c_n are arbitrary coefficients to be determined by the boundary conditions.

3. Total waves of poroelastic half-space and alluvial valley

To solve the boundary conditions at the half-space boundary, the large circle assumption is applied in this work. That is the half-space boundary is approximated as a nearly flat circular boundary centered at o_2 with a radius $r_2 \gg r_1$ (Fig. 2). The curved surface of the large circle is then used as an approximation of the flat surface of the infinite half-space. It is now obvious that when the radius of the large circle approaches infinity this model approaches that of the circular-arc alluvial valley in the half-space. In this paper, $r_2 = 100r_1$ to insure displacements on the curved surface approaches accurately enough to those of a flat surface in the free field. Convergence of the solutions for various large ratios are tested.

For the scattering of elastic wave by circular-arc alluvial valley with an infinite poroelastic, the total wave field of poroelastic half-space is composed of the incident wave, the reflected wave and the scattered wave [24]

$$\varphi_{\mathrm{I}f}^{(t)} = \varphi_{\mathrm{I}f}^{(i)} + \varphi_{\mathrm{I}f}^{(r)} + \varphi_{f1}^{(s)} + \varphi_{f2}^{(s)} \tag{14a}$$

$$\varphi_{\mathrm{I}s}^{(t)} = \varphi_{\mathrm{I}s}^{(i)} + \varphi_{\mathrm{I}s}^{(r)} + \varphi_{s1}^{(s)} + \varphi_{s2}^{(s)} \tag{14b}$$

$$\boldsymbol{\psi}_{\mathrm{I}}^{(t)} = \boldsymbol{\psi}_{\mathrm{I}}^{(i)} + \boldsymbol{\psi}_{\mathrm{I}}^{(r)} + \boldsymbol{\psi}_{1}^{(s)} + \boldsymbol{\psi}_{2}^{(s)} \tag{14c}$$

By introducing complex variables, the incident plane harmonic waves can be expressed as

$$\varphi_{\mathrm{I}f}^{(i)} = \varphi_{\mathrm{I}f_0} \exp\left\{\frac{\mathrm{i}k_{\mathrm{I}f}}{2}\left[z \exp\left(-\mathrm{i}\left(\frac{\pi}{2} - \beta\right)\right) + \bar{z} \exp\left(\mathrm{i}\left(\frac{\pi}{2} - \beta\right)\right)\right]\right\} \tag{15a}$$

$$\varphi_{\mathrm{I}s}^{(i)} = \varphi_{\mathrm{I}s0} \exp\left\{\frac{\mathrm{i}k_{\mathrm{I}s}}{2}\left[z \exp\left(-\mathrm{i}\left(\frac{\pi}{2} - \beta_0\right)\right) + \bar{z} \exp\left(\mathrm{i}\left(\frac{\pi}{2} - \beta_0\right)\right)\right]\right\} \tag{15b}$$

$$\boldsymbol{\psi}_{\mathrm{I}}^{(i)} = \boldsymbol{\psi}_{\mathrm{I}0} \exp\left\{\frac{\mathrm{i}k_{\mathrm{I}t}}{2}\left[z \exp\left(-\mathrm{i}\left(\frac{\pi}{2} - \gamma\right)\right) + \bar{z} \exp\left(\mathrm{i}\left(\frac{\pi}{2} - \gamma\right)\right)\right]\right\} \tag{15c}$$

where β, β_0, γ are the incident angles of the incident harmonic waves, respectively; $\varphi_{\mathrm{I}f0}$, $\varphi_{\mathrm{I}s0}$, $\boldsymbol{\psi}_{\mathrm{I}0}$ are the amplitude ratios of the three incident waves.

If there is no alluvial valley, the incident wave reflected from the half-space will generate a reflected wave to satisfy the stress free boundary conditions. The reflected wave are expressed as

$$\varphi_{\mathrm{I}f}^{(r)} = A_1 \exp\frac{\mathrm{i}k_{\mathrm{I}f}}{2}\left[z \exp\left(\mathrm{i}\left(\frac{\pi}{2} - \beta'\right)\right) + \bar{z} \exp\left(-\mathrm{i}\left(\frac{\pi}{2} - \beta'\right)\right)\right] \tag{16a}$$

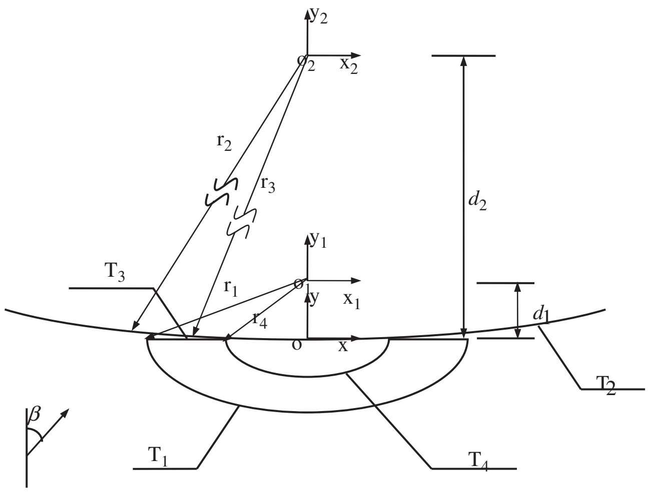

Fig. 2. Approximate model of circular-arc alluvial valley in poroelastic half-space.

$$\varphi_{\mathrm{I}s}^{(r)} = A_2 \exp\frac{\mathrm{i}k_{\mathrm{I}s}}{2}\left[z \exp\left(\mathrm{i}\left(\frac{\pi}{2}-\beta_0'\right)\right) + \bar{z}\exp\left(-\mathrm{i}\left(\frac{\pi}{2}-\beta_0'\right)\right)\right] \tag{16b}$$

$$\boldsymbol{\psi}_{\mathrm{I}}^{(r)} = A_3 \exp\frac{\mathrm{i}k_{\mathrm{I}t}}{2}\left[z \exp\left(\mathrm{i}\left(\frac{\pi}{2}-\gamma'\right)\right) + \bar{z}\exp\left(-\mathrm{i}\left(\frac{\pi}{2}-\gamma'\right)\right)\right] \tag{16c}$$

where A_1, A_2, A_3 are the amplitude ratios; β', β'_0, γ' are the reflected angles, respectively.

The amplitude ratios A_1, A_2 and A_3 are obtained by the straight boundary conditions of poroelastic half-space. Based on the law of Snell, these reflected angles can be expressed as

$$k_{\mathrm{I}f}\sin\beta' = k_{\mathrm{I}s}\sin\beta_0' = k_{\mathrm{I}t}\sin\gamma' \tag{17}$$

In the half-space, because of the presence of both the plane free boundary and the circular-arc alluvial valley, the incident P wave and the reflected P and S waves from the ground surface will be scattered around the valley in the half-space, and the total potentials of harmonic plane P and S waves generated at the valley are represented by $\varphi_{f1}^{(s)}$, $\varphi_{s1}^{(s)}$ and $\boldsymbol{\psi}_1^{(s)}$. The scattered cylindrical waves from the valley will be reflected back into the half-space from the plane free surface. The cylinder vibrations are reflected off the half-space free surface generating new waves represented by $\varphi_{f2}^{(s)}$, $\varphi_{s2}^{(s)}$ and $\boldsymbol{\psi}_2^{(s)}$.

$$\varphi_{\mathrm{I}f}^{(s)} = \varphi_{f1}^{(s)} + \varphi_{f2}^{(s)} = \sum_{i=1}^{2}\sum_{n=-\infty}^{\infty} a_{in}H_n^{(1)}(k_f|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^n \tag{18a}$$

$$\varphi_{\mathrm{I}s}^{(s)} = \varphi_{s1}^{(s)} + \varphi_{s2}^{(s)} = \sum_{i=1}^{2}\sum_{n=-\infty}^{\infty} b_{in}H_n^{(1)}(k_s|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^n \tag{18b}$$

$$\boldsymbol{\psi}_{\mathrm{I}}^{(s)} = \boldsymbol{\psi}_1^{(s)} + \boldsymbol{\psi}_2^{(s)} = \sum_{i=1}^{2}\sum_{n=-\infty}^{\infty} c_{in}H_n^{(1)}(k_t|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^n \tag{18c}$$

where $z_{ij} = z - d_j$ ($j = 1,2$), d_j ($j = 1,2$) is the complex coordinate between the origin of jth circle and the origin of total coordinate system.

For the alluvial valley, the total wave field is composed of the refracted wave and the scattered wave

$$\varphi_{\mathrm{II}f}^{(t)} = \varphi_{\mathrm{II}f}^{(f)} + \varphi_{f3}^{(s)} + \varphi_{f4}^{(s)} \tag{19a}$$

$$\varphi_{\mathrm{II}s}^{(t)} = \varphi_{\mathrm{II}s}^{(f)} + \varphi_{s3}^{(s)} + \varphi_{s4}^{(s)} \tag{19b}$$

$$\boldsymbol{\psi}_{\mathrm{II}}^{(t)} = \boldsymbol{\psi}_{\mathrm{II}}^{(f)} + \boldsymbol{\psi}_3^{(s)} + \boldsymbol{\psi}_4^{(s)} \tag{19c}$$

where

$$\varphi_{\mathrm{II}f}^{(f)} = \sum_{n=-\infty}^{n=+\infty} d_{1n}H_n^{(2)}(k_{\mathrm{II}f}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^n \tag{20a}$$

$$\varphi_{\mathrm{II}s}^{(f)} = \sum_{n=-\infty}^{n=+\infty} e_{1n}H_n^{(2)}(k_{\mathrm{II}s}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^n \tag{20b}$$

$$\boldsymbol{\psi}_{\mathrm{II}}^{(f)} = \sum_{n=-\infty}^{n=+\infty} f_{1n}H_n^{(2)}(k_{\mathrm{II}t}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^n \tag{20c}$$

$$\varphi_{f3}^{(s)} = \sum_{n=-\infty}^{n=+\infty} d_{2n}H_n^{(1)}(k_{\mathrm{II}f}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^n \tag{20d}$$

$$\varphi_{s3}^{(s)} = \sum_{n=-\infty}^{n=+\infty} e_{2n} H_n^{(1)}(k_{\mathrm{II}s}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^n \tag{20e}$$

$$\boldsymbol{\psi}_3^{(s)} = \sum_{n=-\infty}^{n=+\infty} f_{2n} H_n^{(1)}(k_{\mathrm{II}t}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^n \tag{20f}$$

$$\varphi_{f4}^{(s)} = \sum_{n=-\infty}^{n=+\infty} d_{3n} H_n^{(1)}(k_{\mathrm{II}f}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^n \tag{20g}$$

$$\varphi_{s4}^{(s)} = \sum_{n=-\infty}^{n=+\infty} e_{3n} H_n^{(1)}(k_{\mathrm{II}s}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^n \tag{20h}$$

$$\boldsymbol{\psi}_4^{(s)} = \sum_{n=-\infty}^{n=+\infty} f_{3n} H_n^{(1)}(k_{\mathrm{II}t}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^n \tag{20i}$$

where $z_{ij} = z - d_j$ ($j = 3,4$), d_j ($j = 3,4$) is the complex coordinate between the origin of jth circle and the origin of total coordinate system.

4. The expressions of stresses, pore pressures and displacements of poroelastic half-space

If introducing complex variables $z = x + \mathrm{i}y$, $\bar{z} = x - \mathrm{i}y$, the expressions of stresses, pore pressures and displacements of poroelastic half-space can be expressed as

$$u_{\mathrm{I}r} + \mathrm{i}u_{\mathrm{I}\theta} = 2\frac{\partial}{\partial \bar{z}}(\varphi_{\mathrm{I}f}^{(t)} + \varphi_{\mathrm{I}s}^{(t)} - \mathrm{i}\boldsymbol{\psi}_{\mathrm{I}}^{(t)})\exp(-\mathrm{i}\theta) \tag{21a}$$

$$u_{\mathrm{I}r} - \mathrm{i}u_{\mathrm{I}\theta} = 2\frac{\partial}{\partial z}(\varphi_{\mathrm{I}f}^{(t)} + \varphi_{\mathrm{I}s}^{(t)} + \mathrm{i}\boldsymbol{\psi}_{\mathrm{I}}^{(t)})\exp(\mathrm{i}\theta) \tag{21b}$$

$$w_{\mathrm{I}r} + \mathrm{i}w_{\mathrm{I}\theta} = 2\frac{\partial}{\partial \bar{z}}(\eta_{\mathrm{I}1}\varphi_{\mathrm{I}f}^{(t)} + \eta_{\mathrm{I}2}\varphi_{\mathrm{I}s}^{(t)} - \mathrm{i}\alpha_{\mathrm{I}1}\psi_{\mathrm{I}}^{(t)})\exp(-\mathrm{i}\theta) \tag{21c}$$

$$w_{\mathrm{I}r} - \mathrm{i}w_{\mathrm{I}\theta} = 2\frac{\partial}{\partial z}(\eta_{\mathrm{I}1}\varphi_{\mathrm{I}f}^{(t)} + \eta_{\mathrm{I}2}\varphi_{\mathrm{I}s}^{(t)} + \mathrm{i}\alpha_{\mathrm{I}1}\psi_{\mathrm{I}}^{(t)})\exp(\mathrm{i}\theta) \tag{21d}$$

$$\sigma_{\mathrm{I}r} + \sigma_{\mathrm{I}\theta} = -2(\lambda_{\mathrm{I}} + \mu_{\mathrm{I}})(k_{\mathrm{I}f}^2\varphi_{\mathrm{I}f}^{(t)} + k_{\mathrm{I}s}^2\varphi_{\mathrm{I}s}^{(t)}) \tag{21e}$$

$$\sigma_{\mathrm{I}r} + \mathrm{i}\sigma_{\mathrm{I}r\theta} = \alpha_{\mathrm{I}f}\varphi_{\mathrm{I}f}^{(t)} + \alpha_{\mathrm{I}s}\varphi_{\mathrm{I}s}^{(t)} + 4\mu_{\mathrm{I}}\frac{\partial^2}{\partial \bar{z}^2}(\varphi_{\mathrm{I}f}^{(t)} + \varphi_{\mathrm{I}s}^{(t)} - \mathrm{i}\psi_{\mathrm{I}}^{(t)})\exp(-2\mathrm{i}\theta) \tag{21f}$$

$$\sigma_{\mathrm{I}r} - \mathrm{i}\sigma_{\mathrm{I}r\theta} = \alpha_{\mathrm{I}f}\varphi_{\mathrm{I}f}^{(t)} + \alpha_{\mathrm{I}s}\varphi_{\mathrm{I}s}^{(t)} + 4\mu_{\mathrm{I}}\frac{\partial^2}{\partial z^2}(\varphi_{\mathrm{I}f}^{(t)} + \varphi_{\mathrm{I}s}^{(t)} + \mathrm{i}\psi_{\mathrm{I}}^{(t)})\exp(2\mathrm{i}\theta) \tag{21g}$$

$$p_{\mathrm{I}f} = -A_{\mathrm{I}f}k_{\mathrm{I}f}^2\varphi_{\mathrm{I}f}^{(t)} - A_{\mathrm{I}s}k_{\mathrm{I}s}^2\varphi_{\mathrm{I}s}^{(t)} \tag{21h}$$

where

$$\alpha_{\mathrm{I}f} = \alpha_{\mathrm{I}}A_{\mathrm{I}f}k_{\mathrm{I}f}^2 - (\lambda_{\mathrm{I}} + \mu_{\mathrm{I}})k_{\mathrm{I}f}^2, \quad \alpha_{\mathrm{I}s} = \alpha_{\mathrm{I}}A_{\mathrm{I}s}k_{\mathrm{I}s}^2 - (\lambda_{\mathrm{I}} + \mu_{\mathrm{I}})k_{\mathrm{I}s}^2, \quad \eta_{\mathrm{I}1} = \alpha_{\mathrm{I}1} - \alpha_{\mathrm{I}2}A_{\mathrm{I}f}k_{\mathrm{I}f}^2,$$
$$\eta_{\mathrm{I}2} = \alpha_{\mathrm{I}1} - \alpha_{\mathrm{I}2}A_{\mathrm{I}s}k_{\mathrm{I}s}^2, \quad \alpha_{\mathrm{I}1} = \frac{\rho_{\mathrm{I}f}\omega^2}{\beta_{\mathrm{I}1}}, \quad \alpha_{\mathrm{I}2} = -\frac{1}{\beta_{\mathrm{I}1}} \tag{22}$$

In all manipulations, a subscript I is used to denote the parameters of poroelastic half-space.

5. The expressions of stresses, pore pressures and displacements of alluvial valley

If introducing complex variables $z = x+\mathrm{i}y$, $\bar{z} = x - \mathrm{i}y$, the expressions of stresses, pore pressures and displacements of alluvial valley can be expressed as

$$u_{\mathrm{II}r} + \mathrm{i}u_{\mathrm{II}\theta} = 2\frac{\partial}{\partial \bar{z}}(\varphi_{\mathrm{II}f}^{(t)} + \varphi_{\mathrm{II}s}^{(t)} - \mathrm{i}\psi_{\mathrm{II}}^{(t)})\exp(-\mathrm{i}\theta) \tag{23a}$$

$$u_{\mathrm{II}r} - \mathrm{i}u_{\mathrm{II}\theta} = 2\frac{\partial}{\partial z}(\varphi_{\mathrm{II}f}^{(t)} + \varphi_{\mathrm{II}s}^{(t)} + \mathrm{i}\psi_{\mathrm{II}}^{(t)})\exp(\mathrm{i}\theta) \tag{23b}$$

$$w_{\mathrm{II}r} + \mathrm{i}w_{\mathrm{II}\theta} = 2\frac{\partial}{\partial \bar{z}}(\eta_{\mathrm{II}1}\varphi_{\mathrm{II}f}^{(t)} + \eta_{\mathrm{II}2}\varphi_{\mathrm{II}s}^{(t)} - \mathrm{i}\alpha_{\mathrm{II}1}\psi_{\mathrm{II}}^{(t)})\exp(-\mathrm{i}\theta) \tag{23c}$$

$$w_{\mathrm{II}r} - \mathrm{i}w_{\mathrm{II}\theta} = 2\frac{\partial}{\partial z}(\eta_{\mathrm{II}1}\varphi_{\mathrm{II}f}^{(t)} + \eta_{\mathrm{II}2}\varphi_{\mathrm{II}s}^{(t)} + \mathrm{i}\alpha_{\mathrm{II}1}\psi_{\mathrm{II}}^{(t)})\exp(\mathrm{i}\theta) \tag{23d}$$

$$\sigma_{\mathrm{II}r} + \sigma_{\mathrm{II}\theta} = -2(\lambda_{\mathrm{II}} + \mu_{\mathrm{II}})(k_{\mathrm{II}f}^2\varphi_{\mathrm{II}f}^{(t)} + k_{\mathrm{II}f}^2\varphi_{\mathrm{II}f}^{(t)}) \tag{23e}$$

$$\sigma_{\mathrm{II}r} + \mathrm{i}\sigma_{\mathrm{II}r\theta} = \alpha_{\mathrm{II}f}\varphi_{\mathrm{II}f}^{(t)} + \alpha_{\mathrm{II}s}\varphi_{\mathrm{II}s}^{(t)} + 4\mu_{\mathrm{II}}\frac{\partial^2}{\partial \bar{z}^2}(\varphi_{\mathrm{II}f}^{(t)} + \varphi_{\mathrm{II}s}^{(t)} - \mathrm{i}\psi_{\mathrm{II}}^{(t)})\exp(-2\mathrm{i}\theta) \tag{23f}$$

$$\sigma_{\mathrm{II}r} - \mathrm{i}\sigma_{\mathrm{II}r\theta} = \alpha_{\mathrm{II}f}\varphi_{\mathrm{II}f}^{(t)} + \alpha_{\mathrm{II}s}\varphi_{\mathrm{II}s}^{(t)} + 4\mu_{\mathrm{II}}\frac{\partial^2}{\partial z^2}(\varphi_{\mathrm{II}f}^{(t)} + \varphi_{\mathrm{II}s}^{(t)} + \mathrm{i}\psi_{\mathrm{II}}^{(t)})\exp(2\mathrm{i}\theta) \tag{23g}$$

$$p_{\mathrm{II}f} = -A_{\mathrm{II}f}k_{\mathrm{II}f}^2\varphi_{\mathrm{II}f}^{(t)} - A_{\mathrm{II}s}k_{\mathrm{II}s}^2\varphi_{\mathrm{II}s}^{(t)} \tag{23h}$$

where

$$\alpha_{\mathrm{II}f} = \alpha_{\mathrm{II}}A_{\mathrm{II}f}k_{\mathrm{II}f}^2 - (\lambda_{\mathrm{II}} + \mu_{\mathrm{II}})k_{\mathrm{II}f}^2, \quad \alpha_{\mathrm{II}s} = \alpha_{\mathrm{II}}A_{\mathrm{II}s}k_{\mathrm{II}s}^2 - (\lambda_{\mathrm{II}} + \mu_{\mathrm{II}})k_{\mathrm{II}s}^2, \quad \eta_{\mathrm{II}1} = \alpha_{\mathrm{II}1} - \alpha_{\mathrm{II}2}A_{\mathrm{II}f}k_{\mathrm{II}f}^2,$$
$$\eta_{\mathrm{II}2} = \alpha_{\mathrm{II}1} - \alpha_{\mathrm{II}2}A_{\mathrm{II}s}k_{\mathrm{II}s}^2, \quad \alpha_{\mathrm{II}1} = \frac{\rho_{\mathrm{II}f}\omega^2}{\beta_{\mathrm{II}1}}, \quad \alpha_{\mathrm{II}2} = -\frac{1}{\beta_{\mathrm{II}1}} \tag{24}$$

In all manipulations, a subscript II is used to denote the parameters of alluvial valley.

6. The boundary value problems

The boundary conditions of this problem include the continuous conditions at the interface between the valley and the half-space, and the zero stress at the free ground surface within the valley and the half-space out of the valley. The continuous conditions at the interface can be written as

$$\sigma_{\mathrm{I}r} - \mathrm{i}\sigma_{\mathrm{I}r\theta} = \sigma_{\mathrm{II}r} - \mathrm{i}\sigma_{\mathrm{II}r\theta} \tag{25a}$$

$$\sigma_{\mathrm{I}r} + \mathrm{i}\sigma_{\mathrm{I}r\theta} = \sigma_{\mathrm{II}r} + \mathrm{i}\sigma_{\mathrm{II}r\theta} \tag{25b}$$

$$u_{\mathrm{I}r} - \mathrm{i}u_{\mathrm{I}\theta} = u_{\mathrm{II}r} - \mathrm{i}u_{\mathrm{II}\theta} \tag{25c}$$

$$u_{\mathrm{I}r} + \mathrm{i}u_{\mathrm{I}\theta} = u_{\mathrm{II}r} + \mathrm{i}u_{\mathrm{II}\theta} \tag{25d}$$

$$w_{\mathrm{I}r} - \mathrm{i}w_{\mathrm{I}\theta} = w_{\mathrm{II}r} - \mathrm{i}w_{\mathrm{II}\theta} \tag{25e}$$

$$w_{\mathrm{I}r} + \mathrm{i}w_{\mathrm{I}\theta} = w_{\mathrm{II}r} + \mathrm{i}w_{\mathrm{II}\theta} \tag{25f}$$

The zero stress boundary conditions at the free ground surface within the half-space out of the valley can be expressed as

$$\sigma_{\mathrm{I}r} - \mathrm{i}\sigma_{\mathrm{I}r\theta} = 0 \tag{26a}$$

$$\sigma_{\mathrm{I}r} + \mathrm{i}\sigma_{\mathrm{I}r\theta} = 0 \tag{26b}$$

For permeable boundary condition of half-space, the pore pressure should vanish

$$p_{\mathrm{I}f} = -A_{\mathrm{I}f}k_{\mathrm{I}f}^2\varphi_{\mathrm{I}f}^{(t)} - A_{\mathrm{I}s}k_{\mathrm{I}s}^2\varphi_{\mathrm{I}s}^{(t)} = 0 \tag{27}$$

For impermeable boundary condition of half-space, the normal displacement of the fluid relative to the solid skeleton should vanish

$$w_{\mathrm{I}r} = \frac{\partial}{\partial z}(\eta_{\mathrm{I}1}\varphi_{\mathrm{I}f}^{(t)} + \eta_{\mathrm{I}2}\varphi_{\mathrm{I}s}^{(t)} + \mathrm{i}\alpha_{\mathrm{I}1}\boldsymbol{\psi}_{\mathrm{I}}^{(t)})\exp(\mathrm{i}\theta) + \frac{\partial}{\partial\bar{z}}(\eta_{\mathrm{I}1}\varphi_{\mathrm{I}f}^{(t)} + \eta_{\mathrm{I}2}\varphi_{\mathrm{I}s}^{(t)} - \mathrm{i}\alpha_{\mathrm{I}1}\psi_{\mathrm{I}}^{(t)})\exp(-\mathrm{i}\theta) = 0 \tag{28}$$

The zero stress boundary conditions at the free ground surface within the alluvial valley can be expressed as

$$\sigma_{\mathrm{II}r} - \mathrm{i}\sigma_{\mathrm{II}r\theta} = 0 \tag{29a}$$

$$\sigma_{\mathrm{II}r} + \mathrm{i}\sigma_{\mathrm{II}r\theta} = 0 \tag{29b}$$

For permeable boundary condition of alluvial valley, the pore pressure should vanish

$$p_{\mathrm{II}f} = -A_{\mathrm{II}f}k_{\mathrm{II}f}^2\varphi_{\mathrm{II}f}^{(t)} - A_{\mathrm{II}s}k_{\mathrm{II}s}^2\varphi_{\mathrm{II}s}^{(t)} = 0 \tag{30}$$

For impermeable boundary condition of alluvial valley, the normal displacement of the fluid relative to the solid skeleton should vanish

$$\begin{aligned} w_{\mathrm{II}r} = {} & \frac{\partial}{\partial z}(\eta_{\mathrm{II}1}\varphi_{\mathrm{II}f}^{(t)} + \eta_{\mathrm{II}2}\varphi_{\mathrm{II}s}^{(t)} + \mathrm{i}\alpha_{\mathrm{II}1}\psi_{\mathrm{II}}^{(t)})\exp(\mathrm{i}\theta) \\ & + \frac{\partial}{\partial\bar{z}}(\eta_{\mathrm{II}1}\varphi_{\mathrm{II}f}^{(t)} + \eta_{\mathrm{II}2}\varphi_{\mathrm{II}s}^{(t)} - \mathrm{i}\alpha_{\mathrm{II}1}\psi_{\mathrm{II}}^{(t)})\exp(-\mathrm{i}\theta) = 0 \end{aligned} \tag{31}$$

Applying the continuous conditions at the interface between the valley and half-space, and the boundary conditions at the free surface within the valley and half-space out of the valley, substituting the resulted potential functions of the half-space and alluvial valley into above equations, the constants a_{1n}, b_{1n}, c_{1n}, a_{2n}, b_{2n}, c_{2n}, d_{1n}, e_{1n}, f_{1n}, d_{2n}, e_{2n}, f_{2n}, d_{3n}, e_{3n}, f_{3n} can be determined. It should be pointed out that the above equations are all in infinite sums, therefore, the system of equations must be solved by truncating the infinite terms into the finite terms. The number of terms included in the calculation is 12 to reach the required accuracy. The solution courses of (25–31) are given in detail in Appendix A.

7. Numerical results and discussions

Dynamic stress concentration factor σ^* is defined as the ratio of the tangential effective stress along the boundary of the cavity to the normal effective stress

(1) Outer boundary of alluvial valley

$$\sigma_1^* = \frac{\sigma'_{\mathrm{I}\theta}}{\sigma'_0} \tag{32}$$

where

$$\begin{aligned} \sigma'_{\mathrm{I}\theta} = {} & -(\lambda_{\mathrm{I}} + \mu_{\mathrm{I}})(k_{\mathrm{I}f}^2\varphi_{\mathrm{I}f1}^{(t)} + k_{\mathrm{I}s}^2\varphi_{\mathrm{I}s1}^{(t)}) - 2\mu_{\mathrm{I}}\frac{\partial^2}{\partial z^2}(\varphi_{\mathrm{I}f1}^{(t)} + \varphi_{\mathrm{I}s1}^{(t)} + \mathrm{i}\boldsymbol{\psi}_{\mathrm{I}1}^{(t)})\exp(2\mathrm{i}\theta) \\ & - 2\mu_{\mathrm{I}}\frac{\partial^2}{\partial\bar{z}^2}(\varphi_{\mathrm{I}f1}^{(t)} + \varphi_{\mathrm{I}s1}^{(t)} - \mathrm{i}\boldsymbol{\psi}_{\mathrm{I}1}^{(t)})\exp(-2\mathrm{i}\theta) - \alpha_{\mathrm{I}}p_{f\mathrm{I}} \end{aligned} \tag{33a}$$

$$\sigma'_0 = -(\lambda_{\mathrm{I}} + 2\mu_{\mathrm{I}})k_{\mathrm{I}f}^2\varphi_{\mathrm{I}f0} - \alpha_{\mathrm{I}}p_{f0} \tag{33b}$$

(2) Inner boundary of alluvial valley

$$\sigma_2^* = \frac{\sigma'_{\mathrm{II}\theta}}{\sigma'_0} \tag{34}$$

where

$$\sigma'_{\mathrm{II}\theta} = -2(\lambda_{\mathrm{II}} + \mu_{\mathrm{II}})(k^2_{\mathrm{II}f}\varphi^{(t)}_{\mathrm{II}f} + k^2_{\mathrm{II}s}\varphi^{(t)}_{\mathrm{II}s}) - \alpha_{\mathrm{II}}p_{f\mathrm{II}} \tag{35}$$

For the case of impermeable condition, the pore pressure concentration factor is defined as

$$p^* = \frac{p_f}{p_{f0}} \tag{36}$$

where

$$p_{f0} = -A_f k^2_{\mathrm{I}f}\varphi_{f0} \tag{37}$$

In this paper, a semi-analytical solution has been developed for the scattering of plane wave by circular-arc alluvial valley in poroelastic half-space. The effects of frequency, angle of incidence, porosity, and thickness of alluvial valley on the dynamic response will be discussed.

The material parameters for the poroelastic half-space are: $\rho_{\mathrm{I}s} = 2750\,\mathrm{kg/m^3}$; $\rho_{\mathrm{I}f} = 1000\,\mathrm{kg/m^3}$; $n_{\mathrm{I}} = 0.45$; $\mu_{\mathrm{I}} = 2.6 \times 10^7\,\mathrm{Pa}$; $v_{\mathrm{I}} = 0.3$; $\alpha_{\mathrm{I}} = 0.999$; $M_{\mathrm{I}} = 1.0 \times 10^8\,\mathrm{Pa}$; $\eta_{\mathrm{I}} = 1.0 \times 10^{-2}\,\mathrm{Pa\,s}$; $k_{d\mathrm{I}} = 1.0 \times 10^{-10}\,\mathrm{m/s}$. The material parameters for the alluvial valley are: $\rho_{\mathrm{II}s} = 2500\,\mathrm{kg/m^3}$; $\rho_{\mathrm{II}f} = 1000\,\mathrm{kg/m^3}$; $n_{\mathrm{II}} = 0.5$; $\mu_{\mathrm{II}} = 1.5 \times 10^7\,\mathrm{Pa}$; $v_{\mathrm{II}} = 0.25$; $\alpha_{\mathrm{II}} = 0.999$; $M_{\mathrm{II}} = 1.0 \times 10^8\,\mathrm{Pa}$; $\eta_{\mathrm{II}} = 1.0 \times 10^{-2}\,\mathrm{Pa\,s}$; $k_{d\mathrm{II}} = 1.0 \times 10^{-10}\,\mathrm{m/s}$. Figs. 3–8 show the distributions of dynamic stresses concentration and pore pressures concentration around the outer and the inner boundary of circular-arc alluvial valley for incidence angle $\beta = 0^\circ$, 30° and thickness of alluvial valley $r_4/r_1 = 0.8$, 0.7. As shown in Figs. 3–8, the stresses and pore pressures amplitudes increase with the increase of frequency. When the parameters of poroelastic half-space are larger than those of valley, stresses amplitudes around the outer boundary are larger than those around the inner boundary. The effects of incidence angle on stresses and pore pressures patterns with the presence of the valley are also clearly different in Figs. 3–8.

Figs. 9–11 shows the distribution of dynamic stresses concentration and pore pressures concentration around the inner and the outer boundary of circular-arc alluvial valley for valley thickness $r_4/r_1 = 0.75$, 0.8, 0.85. As given the parameters of poroelastic half-space and alluvial valley, the dynamics stresses and pore pressures with the change of valley thickness have clearly regulation. When the parameters of poroelastic half-pace are larger than those of valley, stresses amplitudes around the outer boundary are larger than those around the inner boundary. Pore pressures are greatly smaller than stresses. Stresses amplitudes increase with the increase of thickness of the valley, while pore pressures amplitudes decrease with the increase of thickness

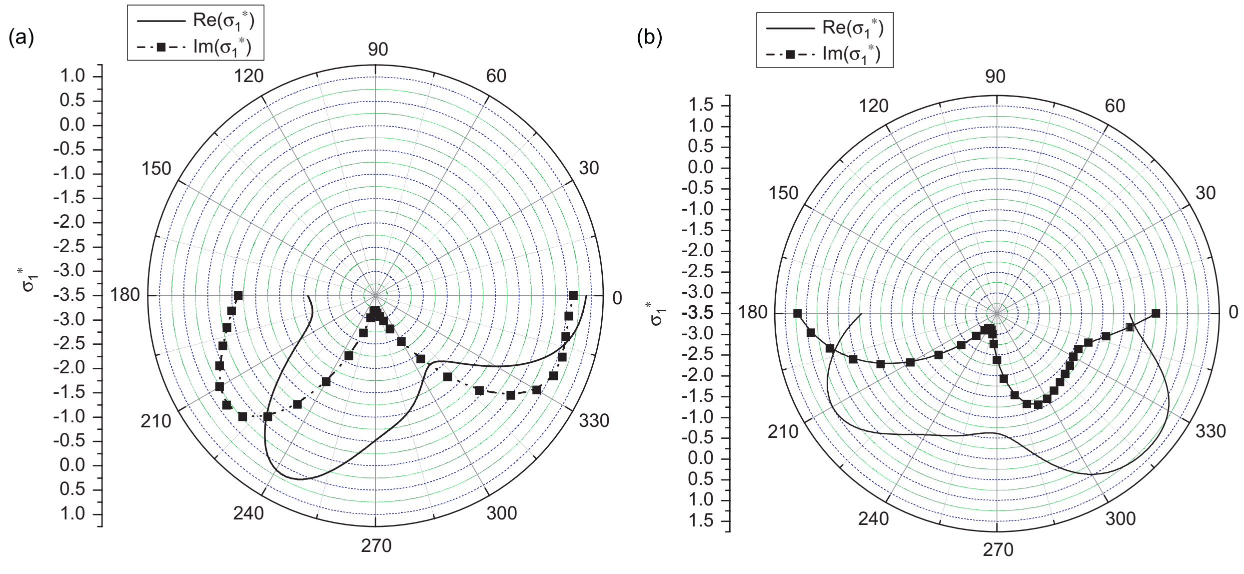

Fig. 3. Distribution of dynamic stresses concentration around the outer boundary of circular-arc alluvial valley ($\beta = 0$, $r_4/r_1 = 0.8$): (a) $\mathrm{Re}(K_f r_1) = 0.25$ and (b) $\mathrm{Re}(K_f r_1) = 1.0$.

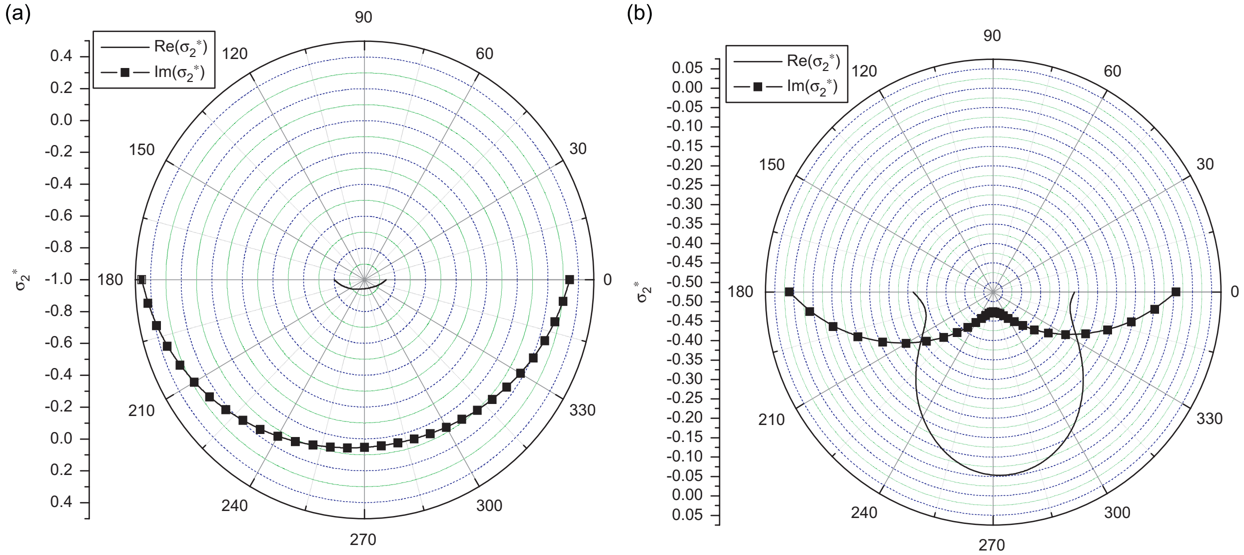

Fig. 4. Distribution of dynamic stresses concentration around the inner boundary of circular-arc alluvial valley ($\beta = 0$, $r_4/r_1 = 0.8$): (a) $\mathrm{Re}(K_f r_1) = 0.25$ and (b) $\mathrm{Re}(K_f r_1) = 1.0$.

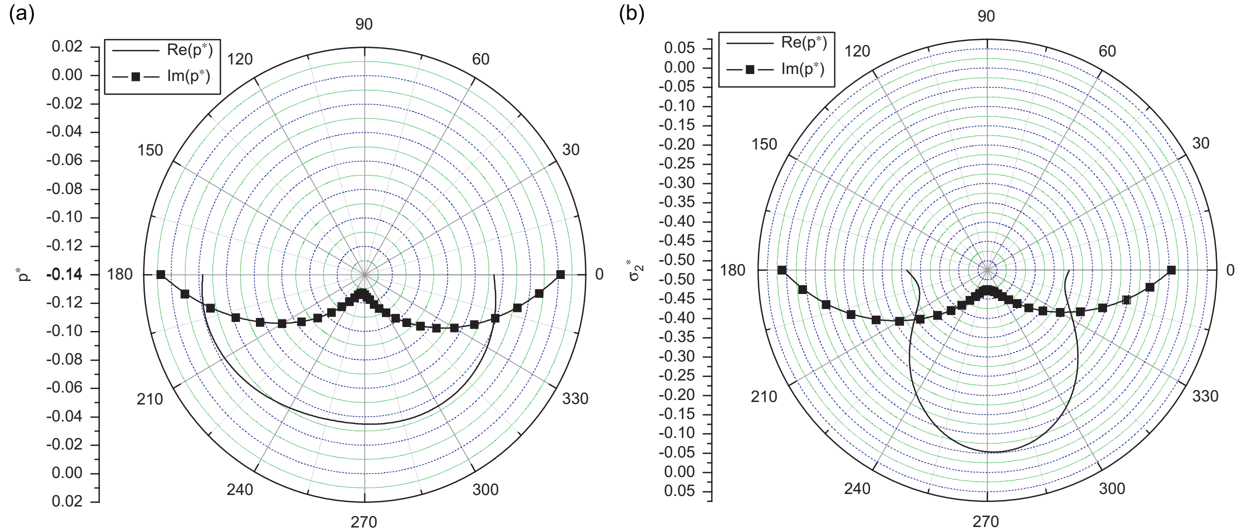

Fig. 5. Distribution of pore pressures concentration around the outer boundary of circular-arc alluvial valley ($\beta = 0$, $r_4/r_1 = 0.8$): (a) $\mathrm{Re}(K_f r_1) = 0.25$ and (b) $\mathrm{Re}(K_f r_1) = 1.0$.

of the valley. When increasing thickness of alluvial valley, stresses amplitudes around the inner and outer boundary of valley may be decreased, while pore pressures amplitudes may be increased. For engineering, the main factor of breakage is the greatest dynamics stresses. So we can increase thickness of alluvial valley to decrease dynamic stresses.

Porosity is one of the important physical parameters of poroelastic half-space. Figs. 12–14 shows the distribution of dynamic stresses concentration and pore pressures concentration around the inner and outer boundary of circular-arc alluvial valley under different porosity $n_{\mathrm{I}}/n_{\mathrm{II}} = 0.9,\ 1.1$. The parameters of poroelastic half-space and alluvial valley are: $\rho_{\mathrm{I}s} = \rho_{\mathrm{II}s} = 2750\,\mathrm{kg/m^3}$; $\rho_{\mathrm{I}f} = \rho_{\mathrm{II}f} = 1000\,\mathrm{kg/m^3}$; $\mu_{\mathrm{I}} = \mu_{\mathrm{II}} = 2.6 \times 10^8\,\mathrm{Pa}$; $v_{\mathrm{I}} = v_{\mathrm{II}} = 0.3$; $\alpha_{\mathrm{I}} = \alpha_{\mathrm{II}} = 0.999$; $M_{\mathrm{I}} = M_{\mathrm{II}} = 1.0 \times 10^8\,\mathrm{Pa}$; $\eta_{\mathrm{I}} = \eta_{\mathrm{II}} = 1.0 \times 10^{-2}\,\mathrm{Pa\,s}$. As shown in Figs. 12–14, the effects of variation of porosity on dynamics stresses and pore pressures are very different. Dynamic stresses amplitudes decrease with the increasing of porosity $n_{\mathrm{I}}/n_{\mathrm{II}}$.

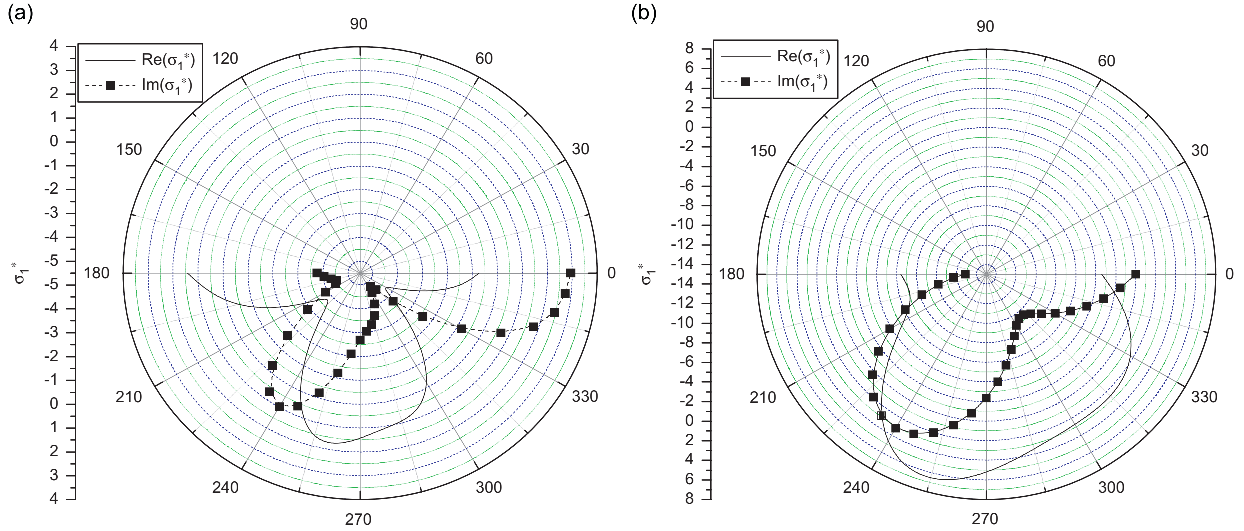

Fig. 6. Distribution of dynamic stresses concentration around the outer boundary of circular-arc alluvial valley ($\beta = 30^\circ$, $r_4/r_1 = 0.7$): (a) $\text{Re}(K_f r_1) = 0.25$ and (b) $\text{Re}(K_f r_1) = 1.0$.

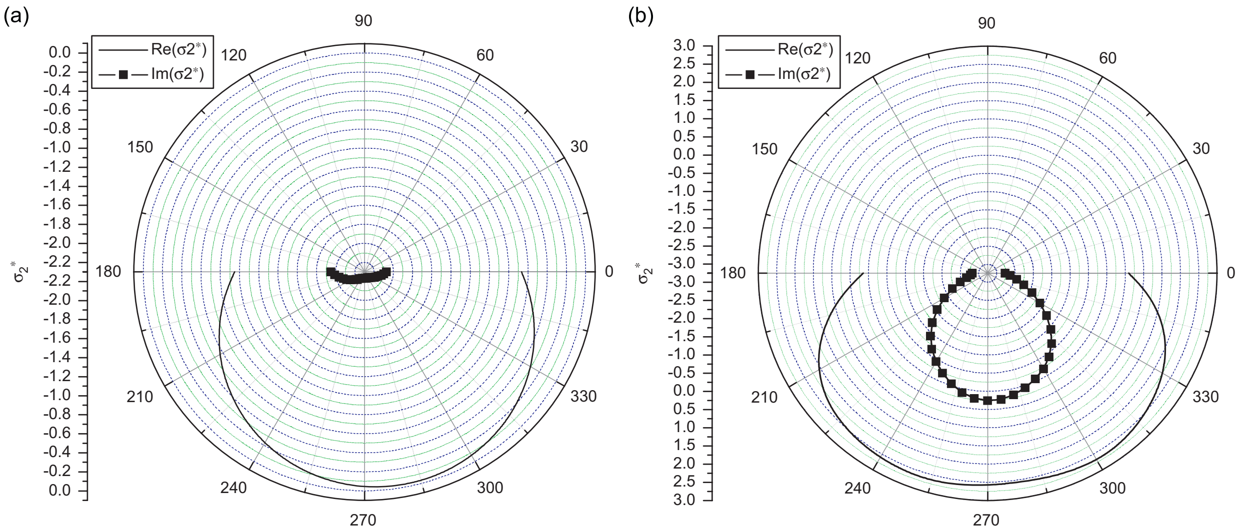

Fig. 7. Distribution of dynamic stresses concentration around the inner boundary of circular-arc alluvial valley ($\beta = 30^\circ$, $r_4/r_1 = 0.7$): (a) $\text{Re}(K_f r_1) = 0.25$ and (b) $\text{Re}(K_f r_1) = 1.0$.

8. Conclusion

Based on Biot's theory and complex variable function method, a new approach for solving the scattering of plane P wave by circular-arc alluvial valley in poroelastic half-space is developed in the paper. The methodology suggested in this paper is more advantageous than the conventional methods, such as eigenfunction expansion method, BEM, FEM for solving wave scattering problems.

The result shows that dynamic stresses amplitudes and pore pressures amplitudes are mainly dependent on angle of incidence, frequency of incident wave, and porosity of soil. The frequency plays an important role in determining stresses and pore pressures patterns. The physical properties of poroelastic medium and alluvial valley have large effects on the scattering of plane P wave by circular-arc alluvial valley. When parameters of poroelastic half-space are greater than those of alluvial valley, dynamic stresses amplitudes increase with thickness of alluvial valley r_4/r_1 increasing, while pore pressures amplitudes decrease with thickness r_4/r_1

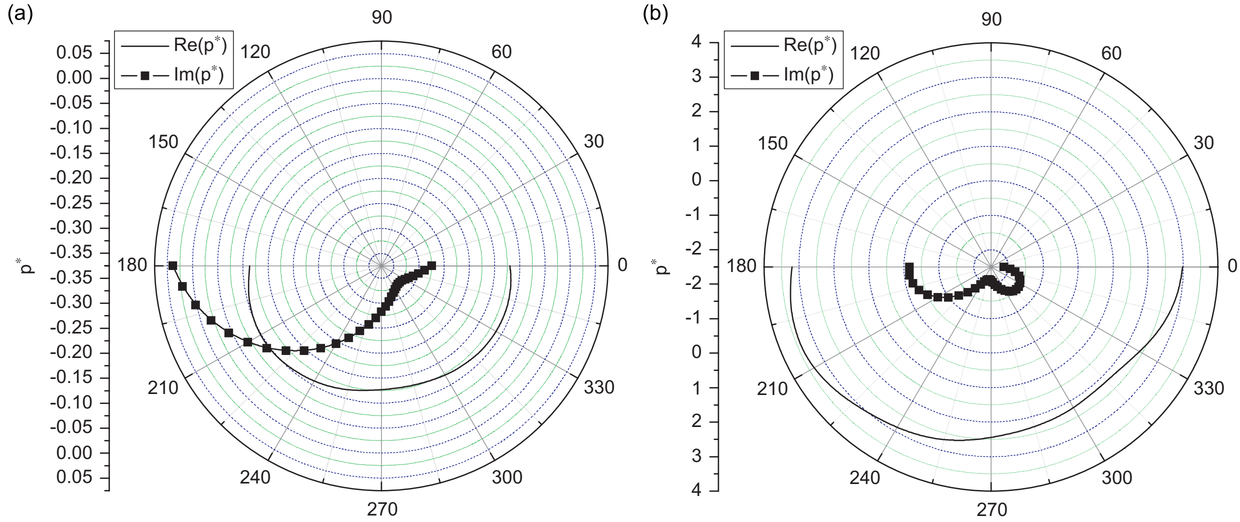

Fig. 8. Distribution of pore pressures concentration around the outer boundary of circular-arc alluvial valley ($\beta = 30^{\circ}$, $r_4/r_1 = 0.7$): (a) $\mathrm{Re}(K_f r_1) = 0.25$ and (b) $\mathrm{Re}(K_f r_1) = 1.0$.

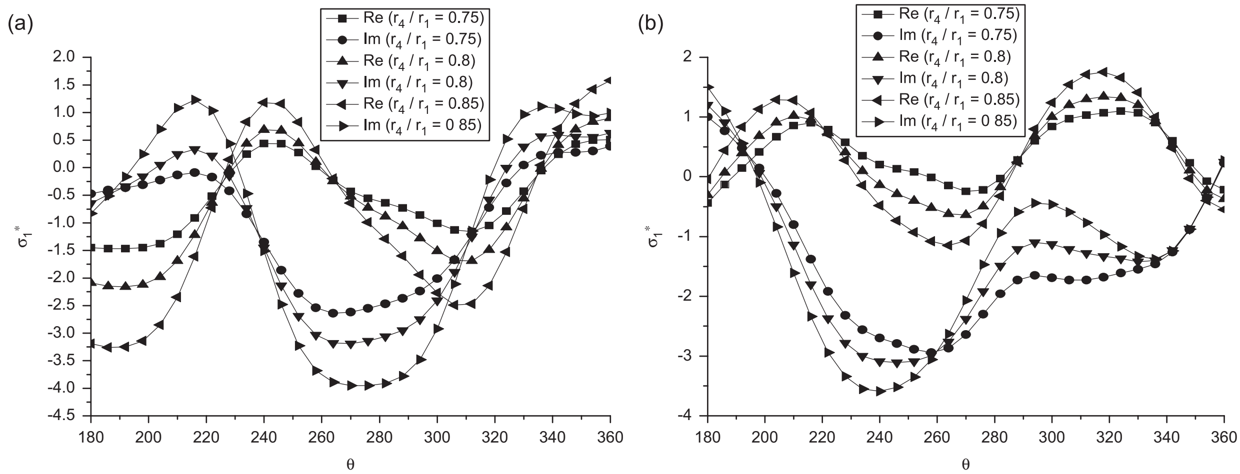

Fig. 9. Distribution of dynamic stresses concentration around the outer boundary of circular-arc alluvial valley with different r_4/r_1: (a) $\mathrm{Re}(K_f r_1) = 0.25$ and (b) $\mathrm{Re}(K_f r_1) = 1.0$.

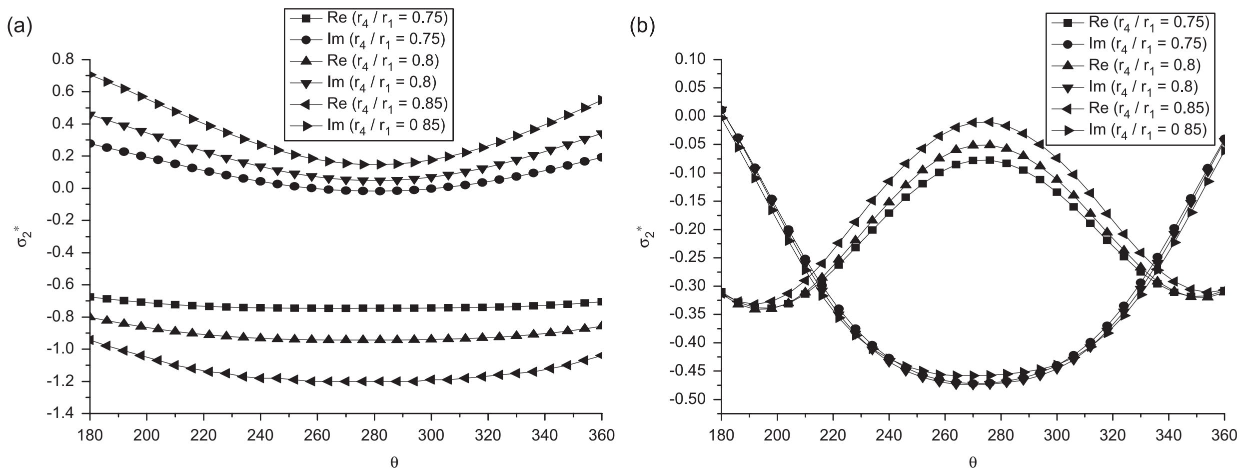

Fig. 10. Distribution of dynamic stresses concentration around the inner boundary of circular-arc alluvial valley with different r_4/r_1: (a) $\mathrm{Re}(K_f r_1) = 0.25$ and (b) $\mathrm{Re}(K_f r_1) = 1.0$.

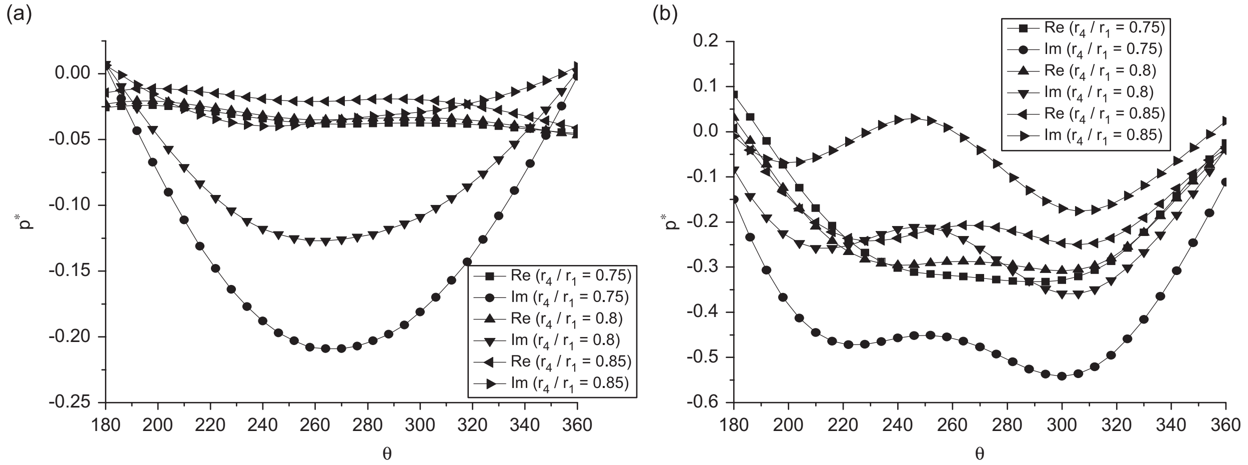

Fig. 11. Distribution of pore pressures concentration around the outer boundary of circular-arc alluvial valley with different r_4/r_1: (a) $\text{Re}(K_f r_1) = 0.25$ and (b) $\text{Re}(K_f r_1) = 1.0$.

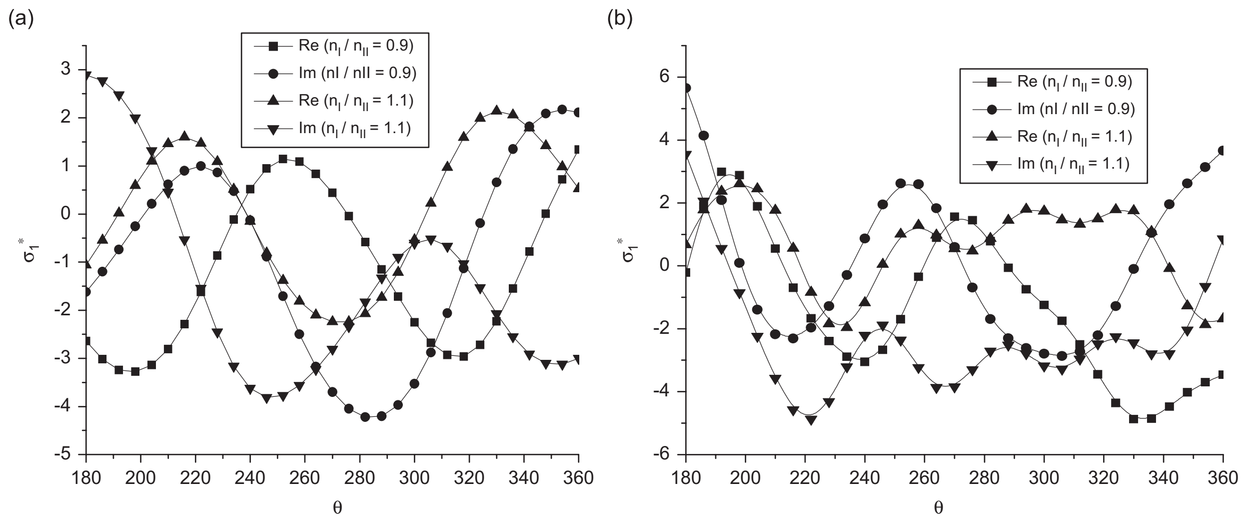

Fig. 12. Distribution of dynamic stresses concentration around the outer boundary of circular-arc alluvial valley with different $n_{\text{I}}/n_{\text{II}}$: (a) $\text{Re}(K_f r_1) = 0.25$ and (b) $\text{Re}(K_f r_1) = 1.0$.

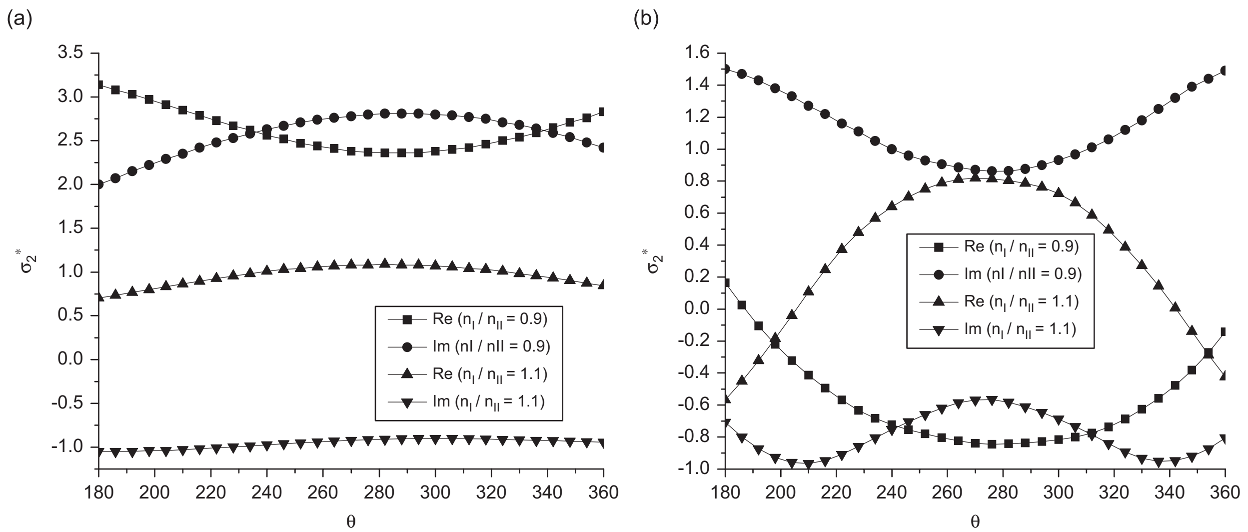

Fig. 13. Distribution of dynamic stresses concentration around the inner boundary of circular-arc alluvial valley with different $n_{\text{I}}/n_{\text{II}}$: (a) $\text{Re}(K_f r_1) = 0.25$ and (b) $\text{Re}(K_f r_1) = 1.0$.

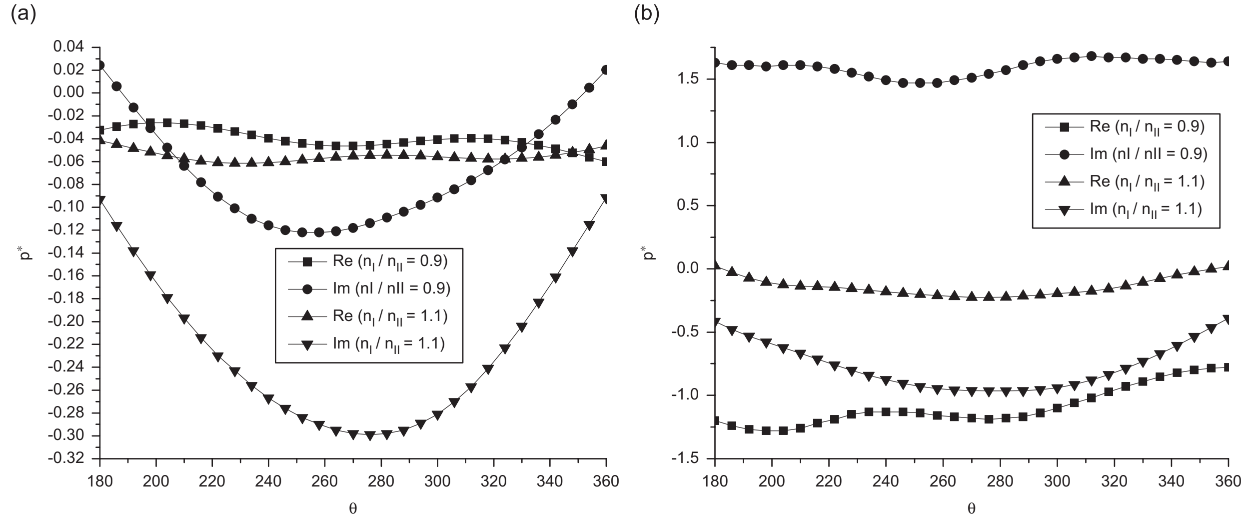

Fig. 14. Distribution of pore pressures concentration around the outer boundary of circular-arc alluvial valley with different $n_{\mathrm{I}}/n_{\mathrm{II}}$: (a) $\mathrm{Re}(K_f r_1) = 0.25$ and (b) $\mathrm{Re}(K_f r_1) = 1.0$.

increasing. Dynamic stresses amplitudes decrease with increasing of porosity $n_{\mathrm{I}}/n_{\mathrm{II}}$. The methodology and analytical solution developed in this paper may analyze the scattering of transient waves by the irregular topography conditions in a finite half-space.

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Appendix A

Substituting Eqs. (21f), (21g), (23f), (23g) into Eqs. (25a), (25b) yields

$$\sum_{p=1}^{3}\sum_{i=1}^{2}\sum_{n=-\infty}^{\infty} E^1_{kpin}x_{pin} + \sum_{m=1}^{3}\sum_{n=-\infty}^{\infty} E^1_{kmn}x_{mn} + \sum_{q=1}^{3}\sum_{j=1}^{2}\sum_{n=-\infty}^{\infty} E^1_{kqjn}x_{qjn} = r^1_k \quad (k=1,2) \tag{A.1}$$

where

$$E^1_{11in} = \alpha_{\mathrm{I}f} H^{(1)}_n(k_{\mathrm{I}f}|z_{ij}|)\left(\frac{z_i}{|z_i|}\right)^n + \mu_{\mathrm{I}} k^2_{\mathrm{I}f} H^{(1)}_{n-2}(k_{\mathrm{I}f}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n-2} \exp(2\mathrm{i}\theta) \tag{A.2}$$

$$E^1_{12in} = \alpha_{\mathrm{I}s} H^{(1)}_n(k_{\mathrm{I}s}|z_i|)\left(\frac{z_i}{|z_i|}\right)^n + \mu_{\mathrm{I}} k^2_{\mathrm{I}s} H^{(1)}_{n-2}(k_{\mathrm{I}s}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n-2} \exp(2\mathrm{i}\theta) \tag{A.3}$$

$$E^1_{13in} = \mathrm{i}\mu_{\mathrm{I}} k^2_{\mathrm{I}t} H^{(1)}_{n-2}(k_{\mathrm{I}t}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n-2} \exp(2\mathrm{i}\theta) \tag{A.4}$$

$$E^1_{11n} = -\alpha_{\mathrm{II}f} H_n^{(2)}(k_{\mathrm{II}f}|r_1\,\mathrm{e}^{\mathrm{i}\theta}|)\left(\frac{r_1\,\mathrm{e}^{\mathrm{i}\theta}}{|r_1\,\mathrm{e}^{\mathrm{i}\theta}|}\right)^n - \mu_{\mathrm{II}} k^2_{\mathrm{II}f} H^{(2)}_{n-2}(k_{\mathrm{II}f}|r_1\,\mathrm{e}^{\mathrm{i}\theta}|)\left(\frac{r_1\,\mathrm{e}^{\mathrm{i}\theta}}{|r_1\,\mathrm{e}^{\mathrm{i}\theta}|}\right)^{n-2} \exp(2\mathrm{i}\theta) \tag{A.5}$$

$$E^1_{12n} = -\alpha_{\mathrm{II}s} H_n^{(2)}(k_{\mathrm{II}s}|r_1\,\mathrm{e}^{\mathrm{i}\theta}|)\left(\frac{r_1\,\mathrm{e}^{\mathrm{i}\theta}}{|r_1\,\mathrm{e}^{\mathrm{i}\theta}|}\right)^n - \mu_{\mathrm{II}} k^2_{\mathrm{II}s} H^{(2)}_{n-2}(k_{\mathrm{II}s}|r_1\,\mathrm{e}^{\mathrm{i}\theta}|)\left(\frac{r_1\,\mathrm{e}^{\mathrm{i}\theta}}{|r_1\,\mathrm{e}^{\mathrm{i}\theta}|}\right)^{n-2} \exp(2\mathrm{i}\theta) \tag{A.6}$$

$$E^1_{13n} = -\mathrm{i}\mu_{\mathrm{II}} k^2_{\mathrm{II}t} H^{(2)}_{n-2}(k_{\mathrm{II}t}|r_1\,\mathrm{e}^{\mathrm{i}\theta}|)\left(\frac{r_1\,\mathrm{e}^{\mathrm{i}\theta}}{|r_1\,\mathrm{e}^{\mathrm{i}\theta}|}\right)^{n-2} \exp(2\mathrm{i}\theta) \tag{A.7}$$

$$E^1_{11jn} = -\alpha_{\mathrm{II}f} H_n^{(1)}(k_{\mathrm{II}f}|z_j|)\left(\frac{z_j}{|z_j|}\right)^n - \mu_{\mathrm{II}} k^2_{\mathrm{II}f} H^{(1)}_{n-2}(k_{\mathrm{II}f}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n-2} \exp(2\mathrm{i}\theta) \tag{A.8}$$

$$E^1_{12jn} = -\alpha_{\mathrm{II}s} H_n^{(1)}(k_{\mathrm{II}s}|z_j|)\left(\frac{z_j}{|z_j|}\right)^n - \mu_{\mathrm{II}} k^2_{\mathrm{II}s} H^{(1)}_{n-2}(k_{\mathrm{II}s}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n-2} \exp(2\mathrm{i}\theta) \tag{A.9}$$

$$E^1_{13jn} = -\mathrm{i}\mu_{\mathrm{II}} k^2_{\mathrm{II}t} H^{(1)}_{n-2}(k_{\mathrm{II}t}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n-2} \exp(2\mathrm{i}\theta) \tag{A.10}$$

$$E^1_{21in} = \alpha_{\mathrm{I}f} H_n^{(1)}(k_{\mathrm{I}f}|z_i|)\left(\frac{z_i}{|z_i|}\right)^n + \mu_{\mathrm{I}} k^2_{\mathrm{I}f} H^{(1)}_{n+2}(k_{\mathrm{I}f}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n+2} \exp(-2\mathrm{i}\theta) \tag{A.11}$$

$$E^1_{22in} = \alpha_{\mathrm{I}s} H_n^{(1)}(k_{\mathrm{I}s}|z_i|)\left(\frac{z_i}{|z_i|}\right)^n + \mu_{\mathrm{I}} k^2_{\mathrm{I}s} H^{(1)}_{n+2}(k_{\mathrm{I}s}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n+2} \exp(-2\mathrm{i}\theta) \tag{A.12}$$

$$E^1_{23in} = \mathrm{i}\mu_{\mathrm{I}} k^2_{\mathrm{I}t} H^{(1)}_{n+2}(k_{\mathrm{I}t}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n+2} \exp(-2\mathrm{i}\theta) \tag{A.13}$$

$$E^1_{21in} = \alpha_{\mathrm{II}f} H_n^{(2)}(k_{\mathrm{II}f}|r_1\,\mathrm{e}^{\mathrm{i}\theta}|)\left(\frac{r_1\,\mathrm{e}^{\mathrm{i}\theta}}{r_1\,\mathrm{e}^{\mathrm{i}\theta}}\right)^n - \mu_{\mathrm{I}} k^2_{\mathrm{II}f} H^{(2)}_{n+2}(k_{\mathrm{II}f}|r_1\,\mathrm{e}^{\mathrm{i}\theta}|)\left(\frac{r_1\,\mathrm{e}^{\mathrm{i}\theta}}{r_1\,\mathrm{e}^{\mathrm{i}\theta}}\right)^{n+2} \exp(-2\mathrm{i}\theta) \tag{A.14}$$

$$E^1_{22in} = -\alpha_{\mathrm{II}s} H_n^{(2)}(k_{\mathrm{II}s}|r_1\,\mathrm{e}^{\mathrm{i}\theta}|)\left(\frac{r_1\,\mathrm{e}^{\mathrm{i}\theta}}{|r_1\,\mathrm{e}^{\mathrm{i}\theta}|}\right)^n - \mu_{\mathrm{II}} k^2_{\mathrm{II}s} H^{(1)}_{n+2}(k_{\mathrm{II}s}|r_1\,\mathrm{e}^{\mathrm{i}\theta}|)\left(\frac{r_1\,\mathrm{e}^{\mathrm{i}\theta}}{|r_1\,\mathrm{e}^{\mathrm{i}\theta}|}\right)^{n+2} \exp(-2\mathrm{i}\theta) \tag{A.15}$$

$$E^1_{23in} = \mathrm{i}\mu_{\mathrm{II}} k^2_{\mathrm{II}t} H^{(1)}_{n+2}(k_{\mathrm{II}t}|r_1\,\mathrm{e}^{\mathrm{i}\theta}|)\left(\frac{r_1\,\mathrm{e}^{\mathrm{i}\theta}}{|r_1\,\mathrm{e}^{\mathrm{i}\theta}|}\right)^{n+2} \exp(-2\mathrm{i}\theta) \tag{A.16}$$

$$E^1_{21jn} = -\alpha_{\mathrm{II}f} H_n^{(1)}(k_{\mathrm{II}f}|z_j|)\left(\frac{z_j}{|z_j|}\right)^n - \mu_{\mathrm{II}} k^2_{\mathrm{II}f} H^{(1)}_{n+2}(k_{\mathrm{II}f}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n+2} \exp(-2\mathrm{i}\theta) \tag{A.17}$$

$$E^1_{22jn} = -\alpha_{\mathrm{II}s} H_n^{(1)}(k_{\mathrm{II}s}|z_j|)\left(\frac{z_j}{|z_j|}\right)^n - \mu_{\mathrm{II}} k^2_{\mathrm{II}s} H^{(1)}_{n+2}(k_{\mathrm{II}s}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n+2} \exp(-2\mathrm{i}\theta) \tag{A.18}$$

$$E^1_{23jn} = \mathrm{i}\mu_{\mathrm{II}} k^2_{\mathrm{II}t} H^{(1)}_{n+2}(k_{\mathrm{II}t}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n+2} \exp(-2\mathrm{i}\theta) \tag{A.19}$$

$$\begin{aligned} r_1^1 = &- \alpha_{\mathrm{I}f}(\varphi^{(i)}_{\mathrm{I}f} + \varphi^{(\gamma)}_{\mathrm{I}f}) - \alpha_{\mathrm{I}s}(\varphi^{(i)}_{\mathrm{I}s} + \varphi^{(\gamma)}_{\mathrm{I}s}) \\ &- 4\mu_{\mathrm{I}} \frac{\partial^2}{\partial z^2}[\varphi^{(i)}_{\mathrm{I}f} + \varphi^{(\gamma)}_{\mathrm{I}f} + \varphi^{(i)}_{\mathrm{I}s} + \varphi^{(\gamma)}_{\mathrm{I}s} + \mathrm{i}(\psi^{(i)}_{\mathrm{I}} + \psi^{(\gamma)}_{\mathrm{I}})] \exp(2\mathrm{i}\theta) \end{aligned} \tag{A.20}$$

$$r_2^1 = -\alpha_{\mathrm{I}f}(\varphi_{\mathrm{I}f}^{(i)} + \varphi_{\mathrm{I}f}^{(\gamma)}) - \alpha_{\mathrm{I}s}(\varphi_{\mathrm{I}s}^{(i)} + \varphi_{\mathrm{I}s}^{(\gamma)}) - 4\mu_{\mathrm{I}}\frac{\partial^2}{\partial \bar{z}^2}[\varphi_{\mathrm{I}f}^{(i)} + \varphi_{\mathrm{I}f}^{(\gamma)} + \varphi_{\mathrm{I}s}^{(i)} + \varphi_{\mathrm{I}s}^{(\gamma)} - \mathrm{i}(\psi_{\mathrm{I}}^{(i)} + \psi_{\mathrm{I}}^{(\gamma)})]\, exp(-2\mathrm{i}\theta) \tag{A.21}$$

where

$$z_i = z_j = r_1\,\mathrm{e}^{\mathrm{i}\theta} \quad (i = j = 1) \tag{A.22}$$

$$z_i = r_1\,\mathrm{e}^{\mathrm{i}\theta} + d - d_1, \quad z_j = r_1\,\mathrm{e}^{\mathrm{i}\theta} + d - d_2 \quad (i = j = 2) \tag{A.23}$$

$$x_{1in} = a_{in}, \quad x_{2in} = b_{in}, \quad x_{3in} = c_{in}, \quad x_{1n} = d_{1n}, \quad x_{2n} = e_{1n}, \quad x_{3n} = f_{1n}, \quad x_{1jn} = d_{jn}, \quad x_{2jn} = e_{jn}, \quad x_{3jn} = f_{jn} \tag{A.24}$$

Multiplying both sides of Eq. (A.1) with $\mathrm{e}^{-\mathrm{i}s\theta}$ and integrating over the interval $[-\pi, \pi]$, we have

$$\sum_{p=1}^{3}\sum_{i=1}^{2}\sum_{n=-\infty}^{\infty} E_{kpin}^{1s} x_{pin} + \sum_{m=1}^{3}\sum_{n=-\infty}^{\infty} E_{kmn}^{1s} x_{mn} + \sum_{q=1}^{3}\sum_{j=1}^{2}\sum_{n=-\infty}^{\infty} E_{kqjn}^{1s} x_{qjn} = r_k^{1s} \quad (s = \pm 0, \pm 1, \ldots) \tag{A.25}$$

where

$$E_{kpin}^{1s} = \frac{1}{2\pi}\int_{-\pi}^{\pi} E_{kpin}^{1}\,\mathrm{e}^{-\mathrm{i}s\theta}\,\mathrm{d}\theta \tag{A.26}$$

$$E_{kmn}^{1s} = \frac{1}{2\pi}\int_{-\pi}^{\pi} E_{kmn}^{1}\,\mathrm{e}^{-\mathrm{i}s\theta}\,\mathrm{d}\theta \tag{A.27}$$

$$E_{kqjn}^{1s} = \frac{1}{2\pi}\int_{-\pi}^{\pi} E_{kqjn}^{1}\,\mathrm{e}^{-\mathrm{i}s\theta}\,\mathrm{d}\theta \tag{A.28}$$

$$r_k^{2s} = \frac{1}{2\pi}\int_{-\pi}^{\pi} r_k^{1}\,\mathrm{e}^{-\mathrm{i}s\theta}\,\mathrm{d}\theta \tag{A.29}$$

Likewise, substituting Eqs. (21a), (21b), (23a), (23b) into Eqs. (25c) and (25d) yields

$$\sum_{p=1}^{3}\sum_{i=1}^{2}\sum_{n=-\infty}^{\infty} E_{kpin}^{2} x_{pin} + \sum_{m=1}^{3}\sum_{n=-\infty}^{\infty} E_{kmn}^{2} x_{mn} + \sum_{q=1}^{3}\sum_{j=1}^{2}\sum_{n=-\infty}^{\infty} E_{kqjn}^{2} x_{qjn} = r_k^{2} \quad (k = 1, 2) \tag{A.30}$$

where

$$E_{11in}^{2} = k_{\mathrm{I}f} H_{n-1}^{(1)}(k_{\mathrm{I}f}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n-1} \tag{A.31}$$

$$E_{12in}^{2} = k_{\mathrm{I}s} H_{n-1}^{(1)}(k_{\mathrm{I}s}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n-1} \tag{A.32}$$

$$E_{13in}^{2} = \mathrm{i}k_{\mathrm{I}t} H_{n-1}^{(1)}(k_{\mathrm{I}t}|z_i|)\left(\frac{z_i}{|z_j|}\right)^{n-1} \tag{A.33}$$

$$E_{11n}^{2} = -k_{\mathrm{II}f} H_{n-1}^{(2)}(k_{\mathrm{II}f}|r_1\,\mathrm{e}^{\mathrm{i}\theta}|)\left(\frac{r_1\,\mathrm{e}^{\mathrm{i}\theta}}{|r_1\,\mathrm{e}^{\mathrm{i}\theta}|}\right)^{n-1} \tag{A.34}$$

$$E_{12n}^{2} = -k_{\mathrm{II}s} H_{n-1}^{(2)}(k_{\mathrm{II}s}|r_1\,\mathrm{e}^{\mathrm{i}\theta}|)\left(\frac{r_1\,\mathrm{e}^{\mathrm{i}\theta}}{|r_1\,\mathrm{e}^{\mathrm{i}\theta}|}\right)^{n-1} \tag{A.35}$$

$$E^2_{13n} = -\mathrm{i}k_{\mathrm{II}t}H^{(2)}_{n-1}(k_{\mathrm{II}t}||r_1\,\mathrm{e}^{\mathrm{i}\theta}||)\left(\frac{|r_1\,\mathrm{e}^{\mathrm{i}\theta}|}{||r_1\,\mathrm{e}^{\mathrm{i}\theta}||}\right)^{n-1} \tag{A.36}$$

$$E^2_{11jn} = -k_{\mathrm{II}f}H^{(1)}_{n-1}(k_{\mathrm{II}f}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n-1} \tag{A.37}$$

$$E^2_{12jn} = -k_{\mathrm{II}s}H^{(1)}_{n-1}(k_{\mathrm{II}s}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n-1} \tag{A.38}$$

$$E^2_{13jn} = -\mathrm{i}k_{\mathrm{II}t}H^{(1)}_{n-1}(k_{\mathrm{II}t}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n-1} \tag{A.39}$$

$$E^2_{21in} = k_{\mathrm{I}f}H^{(1)}_{n+1}(k_{\mathrm{I}f}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n+1} \tag{A.40}$$

$$E^2_{22in} = k_{\mathrm{I}s}H^{(1)}_{n+1}(k_{\mathrm{I}s}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n+1} \tag{A.41}$$

$$E^2_{23in} = -\mathrm{i}k_{\mathrm{I}t}H^{(1)}_{n+1}(k_{\mathrm{I}t}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n+1} \tag{A.42}$$

$$E^2_{21n} = -k_{\mathrm{II}f}H^{(2)}_{n+1}(k_{\mathrm{II}f}|r_1\,\mathrm{e}^{\mathrm{i}\theta}|)\left(\frac{r_1\,\mathrm{e}^{\mathrm{i}\theta}}{|r_1\,\mathrm{e}^{\mathrm{i}\theta}|}\right)^{n+1} \tag{A.43}$$

$$E^2_{22n} = -k_{\mathrm{II}s}H^{(2)}_{n+1}(k_{\mathrm{II}s}|r_1\,\mathrm{e}^{\mathrm{i}\theta}|)\left(\frac{r_1\,\mathrm{e}^{\mathrm{i}\theta}}{|r_1\,\mathrm{e}^{\mathrm{i}\theta}|}\right)^{n+1} \tag{A.44}$$

$$E^2_{23n} = \mathrm{i}k_{\mathrm{II}t}H^{(2)}_{n+1}(k_{\mathrm{II}t}|r_1\,\mathrm{e}^{\mathrm{i}\theta}|)\left(\frac{r_1\,\mathrm{e}^{\mathrm{i}\theta}}{|r_1\,\mathrm{e}^{\mathrm{i}\theta}|}\right)^{n+1} \tag{A.45}$$

$$E^2_{21jn} = -k_{\mathrm{II}f}H^{(1)}_{n+1}(k_{\mathrm{II}f}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n+1} \tag{A.46}$$

$$E^2_{22jn} = -k_{\mathrm{II}s}H^{(1)}_{n+1}(k_{\mathrm{II}s}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n+1} \tag{A.47}$$

$$E^2_{23jn} = \mathrm{i}k_{\mathrm{II}t}H^{(1)}_{n+1}(k_{\mathrm{II}t}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n+1} \tag{A.48}$$

$$r^2_1 = -2\frac{\partial}{\partial z}[\varphi^{(i)}_{\mathrm{I}f} + \varphi^{(r)}_{\mathrm{I}f} + \varphi^{(i)}_{\mathrm{I}s} + \varphi^{(r)}_{\mathrm{I}s} + \mathrm{i}(\psi^{(i)}_{\mathrm{I}} + \psi^{(r)}_{\mathrm{I}})] \tag{A.49}$$

$$r^2_2 = -2\frac{\partial}{\partial \bar{z}}[\varphi^{(i)}_{\mathrm{I}f} + \varphi^{(r)}_{\mathrm{I}f} + \varphi^{(i)}_{\mathrm{I}s} + \varphi^{(r)}_{\mathrm{I}s} - \mathrm{i}(\psi^{(i)}_{\mathrm{I}} + \psi^{(r)}_{\mathrm{I}})] \tag{A.50}$$

Likewise, multiplying both sides of Eq. (A.30) with $\mathrm{e}^{-\mathrm{i}s\theta}$ and integrating over the interval $[-\pi, \pi]$, we have

$$\sum_{p=1}^{3}\sum_{i=1}^{2}\sum_{n=-\infty}^{\infty} E^{2s}_{kpin}x_{pin} + \sum_{m=1}^{3}\sum_{n=-\infty}^{\infty} E^{2s}_{kmn}x_{mn} + \sum_{q=1}^{3}\sum_{j=1}^{2}\sum_{n=-\infty}^{\infty} E^{2s}_{kqjn}x_{qjn} = r^{2s}_k \quad (s = \pm 0, \pm 1\ldots) \tag{A.51}$$

where

$$E_{kpin}^{2s} = \frac{1}{2\pi}\int_{-\pi}^{\pi} E_{kpin}^{2}\,\mathrm{e}^{-\mathrm{i}s\theta}\,\mathrm{d}\theta \tag{A.52}$$

$$E_{kmn}^{2s} = \frac{1}{2\pi}\int_{-\pi}^{\pi} E_{kmn}^{2}\,\mathrm{e}^{-\mathrm{i}s\theta}\,\mathrm{d}\theta \tag{A.53}$$

$$E_{kqjn}^{2s} = \frac{1}{2\pi}\int_{-\pi}^{\pi} E_{kqjn}^{2}\,\mathrm{e}^{-\mathrm{i}s\theta}\,\mathrm{d}\theta \tag{A.54}$$

$$r_{k}^{2s} = \frac{1}{2\pi}\int_{-\pi}^{\pi} r_{k}^{2}\,\mathrm{e}^{-\mathrm{i}s\theta}\,\mathrm{d}\theta \tag{A.55}$$

Likewise, substituting Eqs. (21c), (21d), (23c), (23d) into Eqs. (25e) and (25f) yields

$$\sum_{p=1}^{3}\sum_{i=1}^{2}\sum_{n=-\infty}^{\infty} E_{kpin}^{3}x_{pin} + \sum_{m=1}^{3}\sum_{n=-\infty}^{\infty} E_{kmn}^{3}x_{mn} + \sum_{q=1}^{3}\sum_{j=1}^{2}\sum_{n=-\infty}^{\infty} E_{kqjn}^{3}x_{qjn} = r_{k}^{3} \quad (k = 1, 2) \tag{A.56}$$

where

$$E_{11in}^{3} = \eta_{\mathrm{I}1}k_{\mathrm{I}f}H_{n-1}^{(1)}(k_{\mathrm{I}f}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n-1} \tag{A.57}$$

$$E_{12in}^{3} = \eta_{\mathrm{I}2}k_{\mathrm{I}s}H_{n-1}^{(1)}(k_{\mathrm{I}s}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n-1} \tag{A.58}$$

$$E_{13in}^{3} = \mathrm{i}\alpha_{\mathrm{I}1}k_{\mathrm{I}t}H_{n-1}^{(1)}(k_{\mathrm{I}t}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n-1} \tag{A.59}$$

$$E_{11n}^{3} = -\eta_{\mathrm{II}1}k_{\mathrm{II}f}H_{n-1}^{(2)}(k_{\mathrm{II}f}|r_1\,\mathrm{e}^{\mathrm{i}\theta}|)\left(\frac{r_1\,\mathrm{e}^{\mathrm{i}\theta}}{|r_1\,\mathrm{e}^{\mathrm{i}\theta}|}\right)^{n-1} \tag{A.60}$$

$$E_{12n}^{3} = -\eta_{\mathrm{II}2}k_{\mathrm{II}s}H_{n-1}^{(2)}(k_{\mathrm{II}s}|r_1\,\mathrm{e}^{\mathrm{i}\theta}|)\left(\frac{r_1\,\mathrm{e}^{\mathrm{i}\theta}}{|r_1\,\mathrm{e}^{\mathrm{i}\theta}|}\right)^{n-1} \tag{A.61}$$

$$E_{13n}^{3} = -\mathrm{i}\alpha_{\mathrm{II}1}k_{\mathrm{II}t}H_{n-1}^{(2)}(k_{\mathrm{II}t}|r_1\,\mathrm{e}^{\mathrm{i}\theta}|)\left(\frac{r_1\,\mathrm{e}^{\mathrm{i}\theta}}{|r_1\,\mathrm{e}^{\mathrm{i}\theta}|}\right)^{n-1} \tag{A.62}$$

$$E_{11jn}^{3} = -\eta_{\mathrm{II}1}k_{\mathrm{II}f}H_{n-1}^{(1)}(k_{\mathrm{II}f}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n-1} \tag{A.63}$$

$$E_{12jn}^{3} = -\eta_{\mathrm{II}2}k_{\mathrm{II}s}H_{n-1}^{(1)}(k_{\mathrm{II}s}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n-1} \tag{A.64}$$

$$E_{13jn}^{3} = -\mathrm{i}\alpha_{\mathrm{II}1}k_{\mathrm{II}t}H_{n-1}^{(1)}(k_{\mathrm{II}t}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n-1} \tag{A.65}$$

$$E_{21in}^{3} = \eta_{\mathrm{I}1}k_{\mathrm{I}f}H_{n+1}^{(1)}(k_{\mathrm{I}f}|z_j|)\left(\frac{z_i}{|z_i|}\right)^{n+1} \tag{A.66}$$

$$E_{22in}^{3} = \eta_{\mathrm{I}2}k_{\mathrm{I}s}H_{n+1}^{(1)}(k_{\mathrm{I}s}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n+1} \tag{A.67}$$

$$E^3_{23in} = -\mathrm{i}\alpha_{\mathrm{I}1} k_{\mathrm{I}t} H^{(1)}_{n+1}(k_{\mathrm{I}t}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n+1} \tag{A.68}$$

$$E^3_{21n} = -\eta_{\mathrm{II}1} k_{\mathrm{II}f} H^{(2)}_{n+1}(k_{\mathrm{II}f}|r_1\,\mathrm{e}^{\mathrm{i}\theta}|)\left(\frac{r_1\,\mathrm{e}^{\mathrm{i}\theta}}{|r_1\,\mathrm{e}^{\mathrm{i}\theta}|}\right)^{n+1} \tag{A.69}$$

$$E^3_{22n} = -\eta_{\mathrm{II}2} k_{\mathrm{II}s} H^{(2)}_{n+1}(k_{\mathrm{II}s}|r_1\,\mathrm{e}^{\mathrm{i}\theta}|)\left(\frac{r_1\,\mathrm{e}^{\mathrm{i}\theta}}{|r_1\,\mathrm{e}^{\mathrm{i}\theta}|}\right)^{n+1} \tag{A.70}$$

$$E^3_{23n} = \mathrm{i}\alpha_{\mathrm{II}1} k_{\mathrm{II}t} H^{(2)}_{n+1}(k_{\mathrm{II}t}|r_1\,\mathrm{e}^{\mathrm{i}\theta}|)\left(\frac{r_1\,\mathrm{e}^{\mathrm{i}\theta}}{|r_1\,\mathrm{e}^{\mathrm{i}\theta}|}\right)^{n+1} \tag{A.71}$$

$$E^3_{21jn} = -\eta_{\mathrm{II}1} k_{\mathrm{II}f} H^{(1)}_{n+1}(k_{\mathrm{II}f}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n+1} \tag{A.72}$$

$$E^3_{22jn} = -\eta_{\mathrm{II}2} k_{\mathrm{II}s} H^{(1)}_{n+1}(k_{\mathrm{II}s}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n+1} \tag{A.73}$$

$$E^3_{23jn} = \mathrm{i}\alpha_{\mathrm{II}1} k_{\mathrm{II}t} H^{(1)}_{n+1}(k_{\mathrm{II}t}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n+1} \tag{A.74}$$

$$r^3_1 = -2\frac{\partial}{\partial z}[\eta_{\mathrm{I}1}(\varphi^{(i)}_{\mathrm{I}f} + \varphi^{(r)}_{\mathrm{I}f}) + \eta_{\mathrm{I}2}(\varphi^{(i)}_{\mathrm{I}s} + \varphi^{(r)}_{\mathrm{I}s}) + \mathrm{i}\alpha_{\mathrm{I}1}(\boldsymbol{\psi}^{(i)}_{\mathrm{I}} + \boldsymbol{\psi}^{(r)}_{\mathrm{I}})] \tag{A.75}$$

$$r^3_2 = -2\frac{\partial}{\partial \bar{z}}[\eta_{\mathrm{I}1}(\varphi^{(i)}_{\mathrm{I}f} + \varphi^{(r)}_{\mathrm{I}f}) + \eta_{\mathrm{I}2}(\varphi^{(i)}_{\mathrm{I}s} + \varphi^{(r)}_{\mathrm{I}s}) - \mathrm{i}\alpha_{\mathrm{I}1}(\boldsymbol{\psi}^{(i)}_{\mathrm{I}} + \boldsymbol{\psi}^{(r)}_{\mathrm{I}})] \tag{A.76}$$

Likewise, multiplying both sides of Eq. (A.56) with $\mathrm{e}^{-\mathrm{i}s\theta}$ and integrating over the interval $[-\pi, \pi]$, we have

$$\sum_{p=1}^{3}\sum_{i=1}^{2}\sum_{n=-\infty}^{\infty} E^{3s}_{kpin}x_{pin} + \sum_{m=1}^{3}\sum_{n=-\infty}^{\infty} E^{3s}_{kmn}x_{mn} + \sum_{q=1}^{3}\sum_{j=1}^{2}\sum_{n=-\infty}^{\infty} E^{3s}_{kqjn}x_{qjn} = r^{3s}_k \quad (s = \pm 0, \pm 1\ldots) \tag{A.77}$$

where

$$E^{3s}_{kpin} = \frac{1}{2\pi}\int_{-\pi}^{\pi} E^3_{kpin}\,\mathrm{e}^{-\mathrm{i}s\theta}\,\mathrm{d}\theta \tag{A.78}$$

$$E^{3s}_{kmn} = \frac{1}{2\pi}\int_{-\pi}^{\pi} E^3_{kmn}\,\mathrm{e}^{-\mathrm{i}s\theta}\,\mathrm{d}\theta \tag{A.79}$$

$$E^{3s}_{kqjn} = \frac{1}{2\pi}\int_{-\pi}^{\pi} E^3_{kqjn}\,\mathrm{e}^{-\mathrm{i}s\theta}\,\mathrm{d}\theta \tag{A.80}$$

$$r^{3s}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} r^3_k\,\mathrm{e}^{-\mathrm{i}s\theta}\,\mathrm{d}\theta \tag{A.81}$$

Substituting Eqs. (21f), (21g) into Eqs. (26a), (26b), one has

$$\sum_{p=1}^{3}\sum_{i=1}^{2}\sum_{n=-\infty}^{\infty} E^4_{kpin}x_{pin} = r^4_k \quad (k = 1, 2) \tag{A.82}$$

where

$$E^4_{11in} = \alpha_{\mathrm{I}f} H^{(1)}_n(k_{\mathrm{I}f}|z_i|)\left(\frac{z_i}{|z_i|}\right)^n + \mu_{\mathrm{I}} k^2_{\mathrm{I}f} H^{(1)}_{n-2}(k_{\mathrm{I}f}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n-2}\exp(2\mathrm{i}\theta) \tag{A.83}$$

$$E^4_{12in} = \alpha_{\mathrm{I}s} H^{(1)}_n (k_{\mathrm{I}s}|z_i|)\left(\frac{z_i}{|z_i|}\right)^n + \mu_{\mathrm{I}} k^2_{\mathrm{I}s} H^{(1)}_{n-2}(k_{\mathrm{I}s}|z_{ij}|)\left(\frac{z_i}{|z_i|}\right)^{n-2} \exp(2\mathrm{i}\theta) \tag{A.84}$$

$$E^4_{13in} = \mathrm{i}\mu_{\mathrm{I}} k^2_{\mathrm{I}t} H^{(1)}_{n-2}(k_{\mathrm{I}t}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n-2} \exp(2\mathrm{i}\theta) \tag{A.85}$$

$$E^4_{21in} = \alpha_{\mathrm{I}f} H^{(1)}_n (k_{\mathrm{I}f}|z_i|)\left(\frac{z_i}{|z_i|}\right)^n + \mu_{\mathrm{I}} k^2_{\mathrm{I}f} H^{(1)}_{n+2}(k_{\mathrm{I}f}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n+2} \exp(-2\mathrm{i}\theta) \tag{A.86}$$

$$E^4_{22in} = \alpha_{\mathrm{I}s} H^{(1)}_n (k_{\mathrm{I}s}|z_i|)\left(\frac{z_i}{|z_i|}\right)^n + \mu_{\mathrm{I}} k^2_{\mathrm{I}s} H^{(1)}_{n+2}(k_{\mathrm{I}s}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n+2} \exp(-2\mathrm{i}\theta) \tag{A.87}$$

$$E^4_{13inj} = -\mathrm{i}\mu_{\mathrm{I}} k^2_{\mathrm{I}t} H^{(1)}_{n+2}(k_{\mathrm{I}t}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n+2} \exp(-2\mathrm{i}\theta) \tag{A.88}$$

$$r^4_1 = 0 \tag{A.89}$$

$$r^4_2 = 0 \tag{A.90}$$

$$x_{1in} = a_{in}, \quad x_{2in} = b_{in}, \quad x_{3in} = c_{in} \tag{A.91}$$

$$z_i = r_2\,\mathrm{e}^{\mathrm{i}\theta} + d_2 - d \quad (i = 1) \tag{A.92}$$

$$z_i = r_2\,\mathrm{e}^{\mathrm{i}\theta} \quad (i = 2) \tag{A.93}$$

Likewise, multiplying both sides of Eq. (A.82) with $\mathrm{e}^{-\mathrm{i}s\theta}$ and integrating over the interval $[-\pi, \pi]$, we have

$$\sum_{p=1}^{5}\sum_{i=1}^{2}\sum_{n=-\infty}^{\infty} E^{4s}_{kpin} x_{pin} = r^{4s}_k \quad (k, i = 1, 2) \quad (s = \pm 0, \pm 1, \pm 2, \ldots) \tag{A.94}$$

where

$$E^{4s}_{kpin} = \frac{1}{2\pi}\int_{-\pi}^{\pi} E^4_{kpin}\,\mathrm{e}^{-\mathrm{i}s\theta}\,\mathrm{d}\theta \tag{A.95}$$

$$r^{4s}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} r^4_k\,\mathrm{e}^{-\mathrm{i}s\theta}\,\mathrm{d}\theta \tag{A.96}$$

Substituting Eq. (21h) into Eq. (27) yields

$$\sum_{p=1}^{2}\sum_{i=1}^{2}\sum_{n=-\infty}^{\infty} E^5_{pin} x_{pin} = r^5 \tag{A.97}$$

where

$$E^5_{1in} = -A_{\mathrm{I}f} k^2_{\mathrm{I}f} H^{(1)}_n (k_{\mathrm{I}f}|z_i|)\left(\frac{z_i}{|z_i|}\right)^n \tag{A.98}$$

$$E^5_{2in} = -A_{\mathrm{I}s} k^2_{\mathrm{I}s} H^{(1)}_n (k_{\mathrm{I}s}|z_i|)\left(\frac{z_i}{|z_i|}\right)^n \tag{A.99}$$

$$r^5 = 0 \tag{A.100}$$

Likewise, multiplying both sides of Eq. (A.97) with $e^{-is\theta}$ and integrating over the interval $[-\pi, \pi]$, we have

$$\sum_{p=1}^{2}\sum_{i=1}^{2}\sum_{n=-\infty}^{\infty} E_{pin}^{5s} x_{pin} = r^{5s} \quad (s = \pm 0, \pm 1, \pm 2, \ldots) \tag{A.101}$$

where

$$E_{pin}^{5s} = \frac{1}{2\pi}\int_{-\pi}^{\pi} E_{pin}^{5}\, e^{-is\theta}\, d\theta \tag{A.102}$$

$$r^{5s} = \frac{1}{2\pi}\int_{-\pi}^{\pi} r^{5}\, e^{-is\theta}\, d\theta \tag{A.103}$$

$$\varphi_{f2}^{(t)} = \varphi_{f1}^{(s)} + \varphi_{f2}^{(s)} \tag{A.104}$$

$$\varphi_{s2}^{(t)} = \varphi_{s1}^{(s)} + \varphi_{s2}^{(s)} \tag{A.105}$$

$$\boldsymbol{\psi}_{2}^{(t)} = \boldsymbol{\psi}_{1}^{(s)} + \boldsymbol{\psi}_{2}^{(s)} \tag{A.106}$$

Likewise, substituting Eq. (21c), (21d) into Eq. (28) yields

$$\sum_{p=1}^{3}\sum_{i=1}^{2}\sum_{n=-\infty}^{\infty} E_{pin}^{6} x_{pin} = r^{6} \quad (j = 1, 2) \tag{A.107}$$

where

$$E_{1in}^{6} = \frac{\eta_{I1}k_{If}}{2} H_{n-1}^{(1)}(k_{If}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n-1} \exp(i\theta) - \frac{\eta_{I1}k_{If}}{2} H_{n+1}^{(1)}(k_{If}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n+1} \exp(-i\theta) \tag{A.108}$$

$$E_{2in}^{6} = \frac{\eta_{I2}k_{Is}}{2} H_{n-1}^{(1)}(k_{Is}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n-1} \exp(i\theta) - \frac{\eta_{I2}k_{Is}}{2} H_{n+1}^{(1)}(k_{Is}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n+1} \exp(-i\theta) \tag{A.109}$$

$$E_{3in}^{6} = \frac{i\alpha_{I1}k_{It}}{2} H_{n-1}^{(1)}(k_{It}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n-1} \exp(i\theta) + \frac{i\alpha_{I1}k_{It}}{2} H_{n+1}^{(1)}(k_{It}|z_i|)\left(\frac{z_i}{|z_i|}\right)^{n+1} \exp(-i\theta) \tag{A.110}$$

$$r^{6} = 0 \tag{A.111}$$

Likewise, multiplying both sides of Eq. (A.107) with $e^{-is\theta}$ and integrating over the interval $[-\pi, \pi]$, we have

$$\sum_{p=1}^{3}\sum_{i=1}^{2}\sum_{n=-\infty}^{\infty} E_{pin}^{6s} x_{pin} = r^{6s} \quad (s = \pm 0, \pm 1, \pm 2, \ldots) \tag{A.112}$$

where

$$E_{pin}^{6s} = \frac{1}{2\pi}\int_{-\pi}^{\pi} E_{pin}^{6}\, e^{-is\theta}\, d\theta \tag{A.113}$$

$$r^{6s} = \frac{1}{2\pi}\int_{-\pi}^{\pi} r^{6}\, e^{-is\theta}\, d\theta \tag{A.114}$$

Substituting Eqs. (23f), (23g) into Eqs. (29a), (29b) one obtains

$$\sum_{m=1}^{3}\sum_{n=-\infty}^{\infty} E_{kmnj}^{8} x_{mn} + \sum_{q=1}^{3}\sum_{i=1}^{2}\sum_{n=-\infty}^{\infty} E_{kqinj}^{8} x_{qin} = r_{kj}^{8}, \quad k = 1, 2, \quad j = 3, 4 \tag{A.115}$$

where

$$E^8_{11nj} = \alpha_{\mathrm{II}f} H^{(2)}_n(k_{\mathrm{II}f}|z_j|)\left(\frac{z_j}{z_j}\right)^n + \mu_{\mathrm{II}} k^2_{\mathrm{II}f} H^{(2)}_{n-2}(k_{\mathrm{II}f}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n-2} \exp(2\mathrm{i}\theta) \tag{A.116}$$

$$E^8_{12nj} = \alpha_{\mathrm{II}s} H^{(2)}_n(k_{\mathrm{II}s}|z_j|)\left(\frac{z_j}{|z_j|}\right)^n + \mu_{\mathrm{II}} k^2_{\mathrm{II}s} H^{(2)}_{n-2}(k_{\mathrm{II}s}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n-2} \exp(2\mathrm{i}\theta) \tag{A.117}$$

$$E^8_{13nj} = \mathrm{i}\mu_{\mathrm{II}} k^2_{\mathrm{II}t} H^{(2)}_{n-2}(k_{\mathrm{II}t}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n-2} \exp(2\mathrm{i}\theta) \tag{A.118}$$

$$E^8_{11inj} = \alpha_{\mathrm{II}f} H^{(1)}_n(k_{\mathrm{II}f}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^n + \mu_{\mathrm{II}} k^2_{\mathrm{II}f} H^{(1)}_{n-2}(k_{\mathrm{II}f}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^{n-2} \exp(2\mathrm{i}\theta) \tag{A.119}$$

$$E^8_{12inj} = \alpha_{\mathrm{II}s} H^{(1)}_n(k_{\mathrm{II}s}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^n + \mu_{\mathrm{II}} k^2_{\mathrm{II}s} H^{(1)}_{n-2}(k_{\mathrm{II}s}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^{n-2} \exp(2\mathrm{i}\theta) \tag{A.120}$$

$$E^8_{13inj} = \mathrm{i}\mu_{\mathrm{II}} k^2_{\mathrm{II}t} H^{(1)}_{n-2}(k_{\mathrm{II}t}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^{n-2} \exp(2\mathrm{i}\theta) \tag{A.121}$$

$$E^8_{21nj} = \alpha_{\mathrm{II}f} H^{(2)}_n(k_{\mathrm{II}f}|z_j|)\left(\frac{z_j}{|z_j|}\right)^n + \mu_{\mathrm{II}} k^2_{\mathrm{II}f} H^{(2)}_{n+2}(k_{\mathrm{I}f}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n+2} \exp(-2\mathrm{i}\theta) \tag{A.122}$$

$$E^8_{22nj} = \alpha_{\mathrm{II}s} H^{(2)}_n(k_{\mathrm{II}s}|z_j|)\left(\frac{z_j}{|z_j|}\right)^n + \mu_{\mathrm{II}} k^2_{\mathrm{II}s} H^{(2)}_{n+2}(k_{\mathrm{II}s}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n+2} \exp(-2\mathrm{i}\theta) \tag{A.123}$$

$$E^8_{23nj} = -\mathrm{i}\mu_{\mathrm{II}} k^2_{\mathrm{II}t} H^{(2)}_{n+2}(k_{\mathrm{II}t}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n+2} \exp(-2\mathrm{i}\theta) \tag{A.124}$$

$$E^8_{21inj} = \alpha_{\mathrm{II}f} H^{(1)}_n(k_{\mathrm{II}f}|z_{ij}|)\left(\frac{z_j}{|z_{ij}|}\right)^n + \mu_{\mathrm{II}} k^2_{\mathrm{II}f} H^{(1)}_{n+2}(k_{\mathrm{II}f}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^{n+2} \exp(-2\mathrm{i}\theta) \tag{A.125}$$

$$E^8_{22inj} = \alpha_{\mathrm{II}s} H^{(1)}_n(k_{\mathrm{II}s}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^n + \mu_{\mathrm{II}} k^2_{\mathrm{II}s} H^{(1)}_{n+2}(k_{\mathrm{II}s}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^{n+2} \exp(-2\mathrm{i}\theta) \tag{A.126}$$

$$E^8_{23inj} = -\mathrm{i}\mu_{\mathrm{II}} k^2_{\mathrm{II}t} H^{(1)}_{n+2}(k_{\mathrm{II}t}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^{n+2} \exp(-2\mathrm{i}\theta) \tag{A.127}$$

$$r^8_{13} = 0 \tag{A.128}$$

$$r^8_{23} = 0 \tag{A.129}$$

$$r^8_{14} = 0 \tag{A.130}$$

where

$$z_j = z_{ij} = r_3\,\mathrm{e}^{\mathrm{i}\theta} + d_2 - d_1 \quad (i = 1,\ j = 3) \tag{A.131}$$

$$z_{ij} = r_3\,\mathrm{e}^{\mathrm{i}\theta} \quad (i = 2,\ j = 3) \tag{A.132}$$

$$z_j = z_{ij} = r_4\,\mathrm{e}^{\mathrm{i}\theta} \quad (i = 1,\ j = 4) \tag{A.133}$$

$$z_{ij} = r_4 \,\mathrm{e}^{\mathrm{i}\theta} + d_1 - d_2 \quad (i = 2,\ j = 4) \tag{A.134}$$

$$x_{1in} = a_{in},\ x_{2in} = b_{in},\ x_{3in} = c_{in},\ x_{1n} = d_{1n},\ x_{2n} = e_{1n},\ x_{3n} = f_{1n},\ x_{1in} = d_{in},\ x_{2in} = e_{in},\ x_{3in} = f_{in} \tag{A.135}$$

Likewise, multiplying both sides of Eq. (A.115) with $\mathrm{e}^{-\mathrm{i}s\theta}$ and integrating over the interval $[-\pi,\ \pi]$, we have

$$\sum_{m=1}^{3}\sum_{n=-\infty}^{\infty} E_{kmnj}^{8s} x_{mn} + \sum_{q=1}^{3}\sum_{i=1}^{2}\sum_{n=-\infty}^{\infty} E_{kqinj}^{8s} x_{qin} = r_{kj}^{8s}, \quad k = 1,2, \quad j = 3,4 \tag{A.136}$$

where

$$E_{kmnj}^{8s} = \frac{1}{2\pi}\int_{-\pi}^{\pi} E_{kmnj}^{8}\,\mathrm{e}^{-\mathrm{i}s\theta}\,\mathrm{d}\theta \tag{A.137}$$

$$E_{kqinj}^{8s} = \frac{1}{2\pi}\int_{-\pi}^{\pi} E_{kqinj}^{8}\,\mathrm{e}^{-\mathrm{i}s\theta}\,\mathrm{d}\theta \tag{A.138}$$

$$r_{kj}^{8s} = \frac{1}{2\pi}\int_{-\pi}^{\pi} r_{kj}^{8}\,\mathrm{e}^{-\mathrm{i}s\theta}\,\mathrm{d}\theta \tag{A.139}$$

Substituting Eq. (23h) into Eq. (30) one obtains

$$\sum_{m=1}^{3}\sum_{n=-\infty}^{\infty} E_{mnj}^{9} x_{mn} + \sum_{q=1}^{3}\sum_{i=1}^{2}\sum_{n=-\infty}^{\infty} E_{qinj}^{9} x_{qin} = r_{j}^{9}, \quad j = 3,4 \tag{A.140}$$

where

$$E_{1nj}^{9} = -A_{\mathrm{II}f} k_{\mathrm{II}f}^{2} H_n^{(2)}(k_{\mathrm{II}f}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n} \tag{A.141}$$

$$E_{2nj}^{9} = -A_{\mathrm{II}s} k_{\mathrm{II}s}^{2} H_n^{(2)}(k_{\mathrm{II}s}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n} \tag{A.142}$$

$$E_{1inj}^{9} = -A_{\mathrm{II}f} k_{\mathrm{II}f}^{2} H_n^{(1)}(k_{\mathrm{II}f}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^{n} \tag{A.143}$$

$$E_{2inj}^{9} = -A_{\mathrm{II}s} k_{\mathrm{II}s}^{2} H_n^{(1)}(k_{\mathrm{II}s}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^{n} \tag{A.144}$$

$$r_j^9 = 0 \tag{A.145}$$

Likewise, multiplying both sides of Eq. (A.140) with $\mathrm{e}^{-\mathrm{i}s\theta}$ and integrating over the interval $[-\pi,\ \pi]$, one obtains

$$\sum_{m=1}^{3}\sum_{n=-\infty}^{\infty} E_{mnj}^{9s} x_{mn} + \sum_{q=1}^{3}\sum_{i=1}^{2}\sum_{n=-\infty}^{\infty} E_{qinj}^{9s} x_{qin} = r_{j}^{9s}, \quad j = 3,4 \tag{A.146}$$

where

$$E_{mnj}^{9s} = \frac{1}{2\pi}\int_{-\pi}^{\pi} E_{mnj}^{9}\,\mathrm{e}^{-\mathrm{i}s\theta}\,\mathrm{d}\theta \tag{A.147}$$

$$E_{qinj}^{9s} = \frac{1}{2\pi}\int_{-\pi}^{\pi} E_{qinj}^{9}\,\mathrm{e}^{-\mathrm{i}s\theta}\,\mathrm{d}\theta \tag{A.148}$$

$$r_{j}^{9s} = \frac{1}{2\pi}\int_{-\pi}^{\pi} r_{j}^{9}\,\mathrm{e}^{-\mathrm{i}s\theta}\,\mathrm{d}\theta \tag{A.149}$$

Substituting Eqs. (23c), (23d) into Eq. (31) one obtains

$$\sum_{m=1}^{3}\sum_{n=-\infty}^{\infty} E_{mnj}^{10}x_{mn} + \sum_{q=1}^{3}\sum_{i=1}^{2}\sum_{n=-\infty}^{\infty} E_{qinj}^{10}x_{qin} = r_j^{10}, \quad j = 3,4 \tag{A.150}$$

where

$$E_{1nj}^{10} = \frac{\eta_{\mathrm{II}1}k_{\mathrm{II}f}}{2}H_{n-1}^{(2)}(k_{\mathrm{II}f}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n-1}\exp(\mathrm{i}\theta) - \frac{\eta_{\mathrm{II}1}k_{\mathrm{II}f}}{2}H_{n+1}^{(2)}(k_{\mathrm{II}f}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n+1}\exp(-\mathrm{i}\theta) \tag{A.151}$$

$$E_{2nj}^{10} = \frac{\eta_{\mathrm{II}2}k_{\mathrm{II}s}}{2}H_{n-1}^{(2)}(k_{\mathrm{II}s}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n-1}\exp(\mathrm{i}\theta) - \frac{\eta_{\mathrm{II}2}k_{\mathrm{II}s}}{2}H_{n+1}^{(2)}(k_{\mathrm{II}s}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n+1}\exp(-\mathrm{i}\theta) \tag{A.152}$$

$$E_{3nj}^{10} = \frac{\mathrm{i}\alpha_{\mathrm{II}1}k_{\mathrm{II}t}}{2}H_{n-1}^{(2)}(k_{\mathrm{II}t}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n-1}\exp(\mathrm{i}\theta) - \frac{\mathrm{i}\alpha_{\mathrm{II}1}k_{\mathrm{II}t}}{2}H_{n+1}^{(2)}(k_{\mathrm{II}t}|z_j|)\left(\frac{z_j}{|z_j|}\right)^{n+1}\exp(-\mathrm{i}\theta) \tag{A.153}$$

$$E_{1inj}^{10} = \frac{\eta_{\mathrm{II}1}k_{\mathrm{II}f}}{2}H_{n-1}^{(1)}(k_{\mathrm{II}f}|z_j|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^{n-1}\exp(\mathrm{i}\theta) - \frac{\eta_{\mathrm{II}2}k_{\mathrm{II}f}}{2}H_{n+1}^{(1)}(k_{\mathrm{II}f}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^{n+1}\exp(-\mathrm{i}\theta) \tag{A.154}$$

$$E_{2inj}^{10} = \frac{\eta_{\mathrm{II}2}k_{\mathrm{II}s}}{2}H_{n-1}^{(1)}(k_{\mathrm{II}s}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^{n-1}\exp(\mathrm{i}\theta) - \frac{\eta_{\mathrm{II}2}k_{\mathrm{II}s}}{2}H_{n+1}^{(1)}(k_{\mathrm{II}s}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^{n+1}\exp(-\mathrm{i}\theta) \tag{A.155}$$

$$E_{3inj}^{10} = \frac{\mathrm{i}\alpha_{\mathrm{II}1}k_{\mathrm{II}t}}{2}H_{n-1}^{(1)}(k_{\mathrm{II}t}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^{n-1}\exp(\mathrm{i}\theta) - \frac{\mathrm{i}\alpha_{\mathrm{II}1}k_{\mathrm{II}t}}{2}H_{n+1}^{(1)}(k_{\mathrm{II}t}|z_{ij}|)\left(\frac{z_{ij}}{|z_{ij}|}\right)^{n+1}\exp(-\mathrm{i}\theta) \tag{A.156}$$

$$r_j^{10} = 0 \tag{A.157}$$

Likewise, multiplying both sides of Eq. (A.150) with $e^{-\mathrm{i}s\theta}$ and integrating over the interval $[-\pi, \pi]$, one obtains

$$\sum_{m=1}^{3}\sum_{n=-\infty}^{\infty} E_{mnj}^{10s}x_{mn} + \sum_{q=1}^{3}\sum_{i=1}^{2}\sum_{n=-\infty}^{\infty} E_{qinj}^{10s}x_{qin} = r_j^{10s} \quad (j = 3,4) \quad (s = \pm 0, \pm 1, \pm 2, \ldots) \tag{A.158}$$

where

$$E_{mnj}^{10s} = \frac{1}{2\pi}\int_{-\pi}^{\pi} E_{mnj}^{10}\, e^{-\mathrm{i}s\theta}\, d\theta \tag{A.159}$$

$$E_{qinj}^{10s} = \frac{1}{2\pi}\int_{-\pi}^{\pi} E_{qinj}^{10}\, e^{-\mathrm{i}s\theta}\, d\theta \tag{A.160}$$

$$r_j^{10s} = \frac{1}{2\pi}\int_{-\pi}^{\pi} r_j^{10}\, e^{-\mathrm{i}s\theta}\, d\theta \tag{A.161}$$

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